ARITHMETIC

BY

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ARITHMETIC

FOR THE USE OF SCHOOLS.

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PREFACE.

In the present work, which is the result of more than ten years' experience in teaching, little has been borrowed from other treatises except those ordinary rules which must in all Arithmetics be given in nearly the same words. It is, as the author believes, original in the treatment of the subject, in the methods of arriving at general rules, and in several points not elsewhere noticed.

The brief explanation of Logarithms is intended for the benefit of those who have to use them for practical purposes, and who have not sufficient acquaintance with Algebra to enable them to understand the theorems upon which their construction is based.

The whole of the Examples being original, and many of them new in principle, form an important addition to the stock now available for teaching.

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Errata.

Page 76, Ex. 18, No. 25, *for* 2022·91 *read* 2122·91.
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CHAPTER I.

ABSTRACT NUMBERS.

.RITHMETIC has been frequently defined to be the Science 'Number, and this it certainly is; but in the general accepta-on of the word it includes something more. As a science has for its subject the construction and development of l systems of expressing numbers, or parts of numbers, the cplanation of all the consequences that follow from the loption of these systems, and the investigation of the mutual lations of the numbers themselves. By means of Calculation, hich is usually considered as a part of Arithmetic, these rinciples are applied to practical purposes. Hence Arith-etic includes both a science and an art.

The first part of the science is the construction of systems 'expressing numbers. There have been many such systems, hich may be divided into two main classes, each containing everal varieties. It is not the object of the present work to onsider any other of these than that which is generally used n the world now. This system belongs to a class where the rinciple of 'local value' is recognised in contradistinction to ystems such as the Greek and Roman, where it is not found; nd the particular variety of this class is determined by the act of the number *ten* being chosen as the scale upon which he local value increases or diminishes. For the purposes of xpression we have ten figures or digits, namely 0 signifying othing, and called *nought*, a *cipher*, or *zero*, and 1, 2, 3, 4, 5, , 7, 8, 9, expressing the numbers *one*, a *unit* or *unity*, *two*, *hree*, *four*, *five*, *six*, *seven*, *eight*, *nine*.

The system will be best explained by showing its application to a particular case. Let there be a large heap of corn, and let it be required to count the number of grains. If ten grains be counted and put aside by themselves, and afterwards ten more, and then ten more, and so on as far as possible; then the whole heap will have been separated into a large number of smaller heaps, each containing ten grains, and probably one containing some number less than ten, say *seven*, besides. Now the number of these heaps of ten, though large, is yet not nearly so large as the original number of grains, so that by this separation a great reduction has been effected in the number with which we have to do. The same process may evidently be repeated. Ten of the heaps of ten may be counted and put aside, and then ten more, and ten more, and so on until as the result we have a number of heaps each containing ten tens, and one containing some number less than ten, say *five*, of tens, besides. For ten tens we use the word *hundred*, and therefore the original heap has thus been resolved into some number of hundreds, five tens, and seven. Taking the hundreds, and continuing the process, we should have some tens of hundreds, or *thousands*, and a number less than ten, say *three*, of hundreds, besides. Afterwards, still proceeding in the same manner, suppose that we have a number of tens of thousands, and *four* thousands besides; then a number of hundreds of thousands and *six* tens of thousands besides; then a number of thousands of thousands, or *millions*, and *eight* hundreds of thousands besides. Lastly, suppose that we cannot proceed further in this way, the number of millions being less than ten, suppose *seven*.

Then the whole number of grains in the heap would be expressed as seven millions, eight hundreds of thousands, six tens of thousands, four thousands, three hundreds, five tens, and seven. Let this be written with figures instead of words, and the names of the different heaps be placed above the corresponding figures. Then the number will be—

Millions	Hundreds of thousands	Tens of thousands	Thousands	Hundreds	Tens	Ones, or units
7	8	6	4	3	5	7

Provided that the values of the several figures are remembered, the words written above them may be left out. And the values may easily be remembered by noticing that they increase from the right towards the left, so that each number has a value ten times greater than the same number one place further towards the right, and ten times less than the same number one place further towards the left. The above number would therefore be written 7864357. This, according to the usual practice, would be read 'seven millions, eight hundred and sixty-four thousand, three hundred and fifty-seven,' and comparing this reading with the expression of the number in words previously given, it will be at once seen that certain abbreviations are used. These abbreviations are as follows: firstly, the plural form of thousands and hundreds is not used; secondly, the words eleven, twelve, thirteen, fourteen, fifteen, sixteen, seventeen, eighteen, nineteen, are used in place of ten and 1, ten and 2, &c., as far as ten and 9; thirdly, the words twenty, thirty, forty, fifty, sixty, seventy, eighty, ninety, are used in place of two tens, three tens, &c., as far as nine tens, and after these words the conjunction 'and' is omitted. Taking as examples, numbers of three figures, 453, 218, 907 would be respectively read, four hundred and fifty-three, two hundred and eighteen, nine hundred and seven. If a number consists of four, five, or six figures, the figures after the third from the right should be read in the way just explained, and the word thousand added. Thus, 23896 should be read twenty-three thousand eight hundred and ninety-six; 4702 is four thousand seven hundred and two; 250008 is two hundred and fifty thousand and eight. It should be observed that the conjunction 'and'

is always used immediately before any number from 1 to 99 being mentioned, and that it is not used anywhere else.

It has now been explained how any number of not more than six figures should be read. If a number consist of more than six figures, let it be divided by commas into periods of six figures, commencing from the right hand, and call the second period from the right millions, the third billions, the fourth trillions, and so on to quadrillions, quintillions, &c. As an example, the number of different ways in which the twelve black and twelve white draughtsmen might be arranged on the draught-board is

677794,282450,430456,394720;

and this number is read, six hundred and seventy-seven thousand seven hundred and ninety-four trillions, two hundred and eighty-two thousand four hundred and fifty billions, four hundred and thirty thousand four hundred and fifty-six millions, three hundred and ninety-four thousand, seven hundred and twenty.

As it is most important that the pupil should thoroughly understand the process of *Notation*, or writing down in figures any given number, and the process of *Numeration*, or reading numbers when expressed in figures, examples of both are here given. (Note A.)

Ex. 1.

Express in figures the numbers:—

1. Twenty-nine. Eighty-six. Fifty-seven.

2. Three hundred and fifty-four. Five hundred and nineteen. Six hundred and eight.

3. Two thousand five hundred and eleven. Eight thousand seven hundred and ninety-six. Thirteen thousand and nine.

4. Twenty-four thousand nine hundred and two. Three hundred and eighty-nine thousand seven hundred and forty-six.

5. Fifteen millions, six hundred and forty-three thousand, five hundred and twenty-eight. Seventy-two millions one thousand and four.

6. Three thousand and two billions four hundred millions and seventeen.

Write down in words the numbers:—

7. 56. 73. 85.
8. 725. 913. 804.
9. 4742. 7006. 23589.
10. 51976. 200306.
11. 48,275319. 675,000032.
12. 5,179584,362814. 940,000000,320005.

When numbers are used without reference to any particular objects, as five, twenty-seven, eight thousand and nineteen, they are called *Abstract Numbers*, but when they have such a reference, as for instance, nine men, two pounds, fifty-four sheep, they are called *Concrete Numbers*.

Numbers may be subjected to four different operations in arithmetic, Addition, Subtraction, Multiplication, and Division, and all arithmetical processes, however complicated, can be nothing but combinations of these four simple operations.

The object of *Addition* is to determine what number represents the sum of two or more numbers taken together. If, for example, three persons are in a room, and afterwards two more come in, it will be found that there are then five in the room, and from that we infer that the sum of the numbers 3 and 2 is 5. The foundation, therefore, of addition is the record of a certain number of facts established by experience, and it should always be taught to little children in this way. A child should find out for himself, for example, that 8 and 7 are 15, by putting down upon a slate 8 strokes, then afterwards putting down 7 more, and counting the total number. The number of primary facts to be thus recorded is 90, arising from each of the numbers 1, 2, 3, &c. as far as 10 being added to each of the numbers 1, 2, 3, &c. as far as 9. This series of results is sufficient to enable us to add together any two numbers, however large. For example, let the numbers be 7539 and 9645. Put down either of these under the other, and commence by adding the right hand figures, 5 and 9 are 14. It has been stated that 14 means 10 and 4, and here

$$\begin{array}{r} 7539 \\ \underline{9645} \\ 17184 \end{array}$$

these two numbers are separated, the 4 being put down in the place of units, and 1 carried forward to the place of tens. This 1 carried forward, added to the 4 and the 3 in the place of tens, gives 8. The 6 and 5 in the place of hundreds give 11, that is 1 in the place of hundreds, and 1 carried forward to the place of thousands. Lastly, 1 and 9 are 10, and 10 and 7 are 17.

Although this is the first and the most elementary example of any process in arithmetic, it yet affords an opportunity of explaining what constitute good and bad habits of working. It is very common and most objectionable for a child to accompany the addition with a monotonous repetition, or rather a chant, of the following words :—'Five and nine are fourteen, put down four and carry one ; one and four are five, and three are eight, six and five are eleven, put down one and carry one ; one and nine are ten, and seven are seventeen;' while the numbers to be carried are successively put down and smeared out, somewhere below on the slate or paper. There is much here that is perfectly useless. It would be far better to say nothing aloud, and to 'say to oneself,' as it is called, or register in the memory, by allowing the mind to dwell upon them, not *processes*, but only *results*. Thus, criticising the words given above ; 'five and nine are fourteen' is more than is required to be remembered, the result, fourteen, being all that is wanted; 'put down four and carry one' is something to be done, and not to be talked about, and the 'carrying' is not made easier by the putting down and then smearing out a figure below, or, if made easier, it is at the expense of accustoming the child not to rely upon his memory at all. All that should be registered in the memory is the series of results, 'fourteen, five, eight, eleven, ten, seventeen.' This may appear at first sight a trivial matter, but attention to it greatly facilitates the acquisition of a power of correct and rapid calculation.

When there are more than two numbers to be added

together, the method is precisely the same, but larger numbers have to be remembered while the addition is being performed. In the following example the mind should dwell only on the results, 'three, eleven, seventeen, twenty-one, thirty, thirty-three; eight, fifteen, twenty-four, thirty-one, thirty-seven, thirty-nine; three, four, eight, sixteen, seventeen, twenty-four; six, eight, eleven, thirteen, eighteen, twenty-six.'

$$\begin{array}{r} 8723 \\ 5169 \\ 2874 \\ 3496 \\ 2178 \\ 4053 \\ \hline 26493 \end{array}$$

After adding several numbers together, it is best to try the addition again, beginning at the top and adding downwards. If the results agree, the work is nearly certain to be correct.

In *Subtraction* a smaller number is taken away from a larger, and the remainder is determined. Supposing that five persons are in a room, and two go away, it would be found that there would remain three. Hence we deduce the fact that two from five leaves three. Now, there are two ways of arriving at facts in subtraction; either they may be obtained independently, as, for example, by a child putting down upon paper or a slate a certain number of marks, then rubbing out a given part of them, and counting the remainder; or they may be obtained by deducing them from the facts of Addition. The latter is practically the best. Thus, from the fact that 8 and 7 are 15, we infer that 7 from 15 leaves 8, and that 8 from 15 leaves 7. The 90 elementary facts of addition furnish us with sufficient inferences of this kind to enable us to work out any question in Subtraction. Let it be required then to subtract 53 from 85. Here, putting the 53 under the 85, we say 3 from 5 leaves 2, 5 from 8 leaves 3. Next, to subtract 243 from 527. Here, 3 from 7 leaves 4, and 4 from 2 gives no possible result, so that some alteration must be made in the arrangement of the numbers before the subtraction can be performed. Let the number in the place of tens in the 527 be supposed to be increased by the value of 1 taken away from

$$\begin{array}{r} 85 \\ 53 \\ \hline 32 \end{array}$$

$$\begin{array}{r} 527 \\ 243 \\ \hline 284 \end{array}$$

the place of hundreds. Then the 527 would be 4 hundreds 12 tens and 7. Taking from this 2 hundreds 4 tens and 3, the remainder would be 2 hundreds, 8 tens, and 4, that is to say, 284. The plan that is practically adopted differs slightly from this. Instead of taking the one hundred from the upper number, we consider that it has to be taken away, and for that purpose increase the lower number to that extent. So that in the place of hundreds, instead of supposing the upper figure to be 4 in place of 5, we suppose the lower number to be 3 in place of 2. Hence a full account of all the process gone through would be '3 from 7 leaves 4, 4 from 12 leaves 8, 1 and 2 are 3, and 3 from 5 leaves 2.' The mind, however, need only dwell on the results, 'four, eight, two.' (Note B.)

Since 7 and 8 are 15, and 8 from 15 leaves 7, it follows that the same number being first added to, and then subtracted from another number, the latter remains unchanged. Hence Addition and Subtraction are contrary operations. The signs + called *plus*, and − called *minus*, are used to express these operations; the sign + being placed before a number to be added, the sign − before a number to be subtracted. The sign = called *equals*, or *is equal to*, is placed between two quantities to express the fact of their being equal to one another. As illustrations of the meaning of these signs, the following examples of their employment are given :—

$$17 + 5 - 8 + 3 - 7 + 5 - 9 = 6.$$
$$6 + 8 + 12 + 14 = 7 + 9 + 11 + 13.$$

The following are examples in Addition and Subtraction :—

78964	94318		
65387	285		
24896	4761	67891	83271
57310	56	58427	5485
82743	238	9464	77786
309300	17845		
	117503		

Ex. 2.

Add together the numbers—

1. 237, 45, 623, 19, and 5.
2. 62493, 85721, 154, 718, and 93627.
3. 57641, 92785, 36592, 81276, and 64391.
4. 118637, 4532, 11, 917, and 472895.
5. 26819, 597, 34, 3087, and 62859.
6. 54831, 73649, 85726, 25963, and 78964.

Subtract—

7. 394 from 811.
8. 6785 from 9463.
9. 2537 from 486729.
10. 5895237 from 6000000.
11. 8179 from 43287.
12. 48396 from 59285.

Find the values of—

13. $7+9+14-20+11-19$.
14. $18-2+33-9-7-4$.
15. $64-32-16-8-4-2-1$.

Prove that—

16. $8+10-14+6=11+20-7-14$.
17. $1+8+27-25=10-3+18-5-9$.
18. $19+37-51-5=29-3+45-71$.

The object of *Multiplication* is to find the sum of a certain number of equal numbers added together. Thus, $5 + 5 + 5 + 5 = 20$, and hence four fives added together or 'four times five,' as it is called, are twenty. And in this manner every question in multiplication might be determined by addition. This would, however, frequently involve long and laborious calculations, and it is possible to avoid these by shorter methods. As in Addition and Subtraction there were a number of elementary facts which served as the material for applying these operations to all numbers whatever, so in Multiplication there is a series of facts, commonly called the 'Multiplication Table,' by the aid of which any multiplication whatever may be performed. Although the 'Multiplication Table' is generally given to children as a lesson to be learned by heart, there are strong reasons for considering this an injudicious practice.

child is thus taught to say 7 times 8 are 56, without any clear idea of why this should be the case, or by what means it was found out. It would be better that he should find out each fact in the table for himself by successive acts of addition. Thus 5 and 5 are 10, and therefore twice 5 are 10; 10 and 5 are 15, therefore 3 times 5 are 15; and by successive additions of 5 are obtained 4 times 5 are 20, 5 times 5 are 25, 6 times 5 are 30, and so on. The number of facts required in the elementary series is 100, carrying on the multiplication to 10 times 10, but as the number 12 is often in practice required as a multiplier, the multiplication table is, as a matter of convenience, extended beyond the limits of what is necessary, and includes 144 facts, going as far as 12 times 12.

MULTIPLICATION TABLE.

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

Suppose a child to have constructed for himself correctly one row of the shorter table, that is, the results of the multiplication of the numbers from 1 to 10 by some number of one figure. He is then able to work out examples of the multiplication of any number whatever, however large, by that one figure. For this purpose he will merely have to multiply each figure of the given number successively by the one figure, beginning from the left, and to carry on to the next place the number of tens that may occur in each of these products. As an instance let it be required to multiply 347 by 4. Here 4 times 7 are 28, 8 must be put down in the place of units, and two carried on to the place of tens; 4 times 4 are 16, and 2 are 18, 8 must be put down and 1 carried; 4 times 3 are 12, and 1 are 13.

$$\begin{array}{r} 347 \\ 4 \\ \hline 1388 \end{array}$$

Examples of this kind should be given to a child until he is by practice thoroughly acquainted with the Multiplication Table, and consequently able to multiply rapidly and accurately by any number of one figure.

Before proceeding to consider multiplication by a number consisting of several figures, the following particulars with regard to multiplication in general must be observed.

1. The sign of multiplication is $\times$, which is read *into*, and means that the numbers between which it stands are to be multiplied together.

2. The number to be multiplied is called the *multiplicand*, the number by which it is to be multiplied the *multiplier*, and the result of the multiplication the *product.*

3. In multiplication the order of the numbers multiplied together, or *factors* as they are called, is immaterial. Thus—

$$5 \times 8 = 8 \times 5 \quad 7 \times 3 \times 9 = 3 \times 7 \times 9 \quad 8 \times 4 \times 5 \times 2 \times 6 = 5 \times 6 \times 4 \times 8 \times 2$$

It will be seen hereafter that this principle has some important applications.

4. If a number be multiplied by another, the product by a third, this second product by a fourth, and so on, the final result, which is called the *continued product*, is the same as if

the first number had been at once multiplied by the product of all the others. Thus—

$$8 \times 7 \times 4 \times 3 = 672 \text{ and } 8 \times 84 = 672.$$

5. If a number be separately multiplied by each of several others, and the products added together, the result is the same as if the first number had been multiplied by the sum of all the others. Thus—

$$9 \times 3 = 27 \quad 9 \times 5 = 45 \quad 9 \times 7 = 63 \quad 27 + 45 + 63 = 135$$
$$3 + 5 + 7 = 15 \quad 9 \times 15 = 135.$$

6. It will be found that the product of any number multiplied by 10, is the same number written down with an additional cipher at the right hand side. Thus—

$$4376 \times 10 = 43760.$$

Consequently, multiplication by 100, which is 10×10, may be performed by writing two ciphers; by 1000 by writing three ciphers, &c., to the right of the multiplicand. Thus—

$$745 \times 100 = 74500 \quad 837 \times 10000 = 8370000$$

Let it now be required to multiply a number by another consisting of several figures, as for instance, 8932 by 745. Here let the multiplier be separated into the numbers 5, 40 and 700, and let 8932 be multiplied by each of these separately. $8932 \times 5 = 44660$; $8932 \times 40 = 8932 \times 4 \times 10$, and consequently the multiplication may be effected by multiplying by 4 and writing one cipher at the right-hand side, making 357280; $8932 \times 700 = 6252400$. Let the three products be added together, and the sum will be 6654340. It is usual to write down the multiplicand with the multiplier beneath it, and under them to write the separate products, and under these their sum.

$$\begin{array}{r} 8932 \\ 745 \\ \hline 44660 \\ 357280 \\ 6252400 \\ \hline 6654340 \end{array}$$

As the ciphers added on to the right hand of these products make no difference in the addition, they may be left out,

provided care is taken to place the right-hand figure of each product in the same position of local value as the figure in the multiplier which produced it.

Thus in the annexed example, in the multiplication by the figure 7, the first figure 4 of the product is in the place of tens of thousands, which was the place of the 7. The best way of ensuring that the figures are placed correctly, is to take care that each right-hand figure of a product is in the same column with, and directly under the corresponding figure in the multiplier.

```
      7653292
       807604
-------------
     30613168
   45919752
  53573044
61226336
-------------
6180829232368
```

Should there be ciphers in either the multiplier, the multiplicand, or in both, no notice need be taken of them, till after the other figures have been multiplied, when they may be added on to the product. The multiplicand and multiplier should be so placed that the first *significant figures*, that is figures other than 0, in each are in the same column. The following are examples:

```
  32700        4531        42300
   38           3790         760
-------      --------     --------
 2616         40779        2538
 981         31717        2961
-------     13593         --------
1242600     --------      32148000
            17172490
```

If the multiplier is itself the product of several factors, the multiplication may be performed either by multiplying by these factors successively, or by the multiplier, at one operation. Thus 47328 may be multiplied by 432 in either of the two following ways.

```
   47328              47328
       8                432
--------           --------
  378624              94656
       6             141984
--------            189312
 2271744           --------
       9           20445696
--------
20445696
```

The first of these methods is seldom employed in multiplications such as those that have hitherto been given. It will be seen however farther on, that there are cases in which it should be used.

Ex. 3.

Find the values of—

1. 6893×7; 7849×5; 86237×7.
2. 32781×9; 57346×11; 98375×12.
3. 957826×29; 78462×430.
4. 83785×326; 947139×512.
5. 47953×1084; 62395×7009.
6. 96327×59871; 46893×20508.

Prove that—

7. $85743 \times 168 = 85743 \times 2 \times 7 \times 12$.
8. $94675 \times 320 = 94675 \times 10 \times 8 \times 4$.
9. $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \times 11 \times 12 = 46080 \times 10395$.
10. $4245 \times 6470 = 3882 \times 7075$.
11. $344839 \times 496 = 330832 \times 517$.
12. $2 \times 4 \times 6 \times 8 \times 10 \times 12 = 2 \times 3 \times 4 \times 5 \times 6 \times 64$.

Division is the opposite of multiplication. The object of Multiplication was stated to be to find 'the sum of a certain number of equal numbers added together.' The object of Division is to find how many given equal numbers must be added together to produce a certain sum. How much is the sum of four fives added together? is a question in Multiplication. How many fives must be added together to produce 20? is a question in Division. The facts contained in the Multiplication Table are the groundwork of the process of Division. When it is known that $7 \times 8 = 56$, it is also known that the number of eights that must be added together to produce 56 is seven, or in other words that 8 is contained in 56 seven times. The number to be divided is called the *dividend;* the number by which it is to be divided, the *divisor;* and the result, the *quotient.* The sign of division is either $\div$ placed before the divisor and after the dividend; or a line placed above the divisor and below the dividend; or : which is sometimes used instead of $\div$. Thus $12 \div 4$, $\frac{12}{4}$, $12 : 4$ all mean the same

thing, are each equal to 3, and may all be read 12 *divided by* 4, or 12 *by* 4.

It must be observed that the above definitions of multiplication and division are not complete. They contain as much of the full meaning as it is possible to employ at this part of the subject. It will be seen hereafter that the signification of these words will be extended.

Suppose it now be required to find how often 8 is contained in 59, or, in other words, to divide 59 by 8. We know that $56 \div 8 = 7$ and that $64 \div 8 = 8$. Hence it follows that $59 \div 8$ must be more than 7 and less than 8. The way to express this is to say that the quotient is 7; and that 3, the difference between 56 and 59 has been left undivided, and forms what is termed the ***remainder.***

To divide 34967 by 6. Since $30{,}000 \div 6 = 5{,}000$ and that $36{,}000 \div 6 = 6{,}000$, it follows that the quotient must be between 5,000 and 6,000. Placing 5 therefore as the first left hand figure in the quotient, and subtracting 30,000 from the dividend, there remains 4967 undivided. Now $4800 \div 6 = 800$, and hence, putting 8 as the second figure in the quotient, there remains 167 to be divided; $120 \div 6 = 20$, and $47 \div 6$ gives as quotient 7, and remainder 5. Hence $34967 \div 6 = 5827$, with remainder 5.

```
6)34967(5827       6)34967(5827       6)34967
  30000              30                 -------
  -----              --                  5827  5
   4967              49
   4800              48
   ----              --
    167               16
    120               12
    ---               --
     47                47
     42                42
     --                --
      5                 5
```

The ciphers may be left out if care is taken to keep the figures representing tens, hundreds, &c., in their proper columns all through the process, and the figures of the dividend may be brought down one by one as they are wanted. In the cases where the divisor consists of one

figure only the work may be still farther abbreviated by performing the multiplication and subtraction in the mind, without writing down the results. Thus the whole statement of the work done in the mind would be, 34 by 6 is 5 and 4 over; 49 by 6 is 8 and 1 over; 16 by 6 is 2 and 4 over; 47 by 6 is 7 and 5 over. The work should, however, be performed quietly, without saying anything aloud, and without writing down, and afterwards rubbing out any of the remainders. When the divisor consists of more than one figure the work is generally fully expressed, as in the following instances:—

```
59)783962(13287
   59
   ---
   193
   177
   ---
    169
    118
    ---
     516
     472
     ---
      442
      413
      ---
       29
```

```
489)3617456(7397
    3423
    ----
     1944
     1467
     ----
      4775
      4401
      ----
       3746
       3423
       ----
        323
```

In working out examples of this kind the following particulars should be observed:—

1. The figures will, if placed at regular distances, naturally arrange themselves, so that the figures representing tens, hundreds, thousands, &c., always keep in their proper columns, and form straight rows in the direction of the length of the page. This natural arrangement should not be interfered with by a careless method of writing down the work.

2. For each figure of the dividend brought down there must be a corresponding figure in the quotient. If the divisor is larger than the number to be divided when the figure is brought down, a cipher must be put in the quotient, and then another figure brought down.

3. The product of the divisor and each successive figure of the quotient must never be greater than the number from which it is to be subtracted. If it should be so, the figure in the quotient is too large.

4. The remainder after each subtraction must be less than the divisor. If it should not be so, the figure in the quotient is too small. (Note C.)

Wherever the divisor is a number composed of several factors, we may either divide by the number at once, or by its factors in succession. Thus, if we had to divide 8973048 by 56, it would make no difference in the result whether we divided by 56 at once, or by 8 and then by 7, or by 7 and then by 8. In each case the quotient would be 160237. In this instance there is no remainder, but in general there will be a remainder, and if the dividend is divided by the several factors of the divisor there will be a series of remainders after each division, from which the total remainder must be deduced. As an instance, let it be required to divide 7951 by 96, and let this division be performed in the four following ways:—

$$\begin{array}{r} 96)7951(82 \\ \underline{768} \\ 271 \\ \underline{192} \\ 79 \end{array} \qquad \begin{array}{r} 12)\underline{7951} \\ 8)\underline{\ 662}-7 \\ 82-6 \end{array}\Big\}79 \qquad \begin{array}{r} 8)\underline{7951} \\ 12)\underline{\ 993}-7 \\ 82-9 \end{array}\Big\}79 \qquad \left.\begin{array}{r} 4)\underline{7951} \\ 4)1987-3 \\ 6)\underline{\ 496}-3 \\ 82-4 \end{array}\right\}79$$

The first method has been already fully explained. In the second, let the first dividend be considered as 7951 units, and the second as 662 twelves; then it is clear that there are 6 twelves and 7 units over, and these together make 79. In the third there are 9 eights and 7 units over, making 79 as before. From these instances we may deduce the following general rule for finding the complete remainder after the division of any number by another by means of two factors: *Multiply the second remainder by the first divisor, and add the first remainder.* In the fourth method the

division is performed by means of three factors, and there are three remainders. Here let the first dividend be considered as 7951 units, the second as 1987 fours, the third as 496 sixteens; then there are 4 sixteens, 3 fours, and 3 units over, making together 79. Hence the rule when the number of factors is more than two: *Multiply each remainder by all the previous divisors and take the sum of these products, adding in the first remainder.*

As multiplication and division are contrary operations, either of them furnishes a proof of the other. Thus, to prove an example in multiplication: *Divide the product by the multiplier, and the quotient should be the multiplicand.* To prove an example in division: *Multiply the quotient by the divisor, add the remainder to the product, and the result should be the dividend.*

Ex. 4.

Find the values of—

1. 71936÷5; 82479÷8; 237842÷6.
2. 84135÷12; 71182÷11; 479365+9.
3. 68732÷13; 85496÷17; 124967÷26.
4. 837562+47; 984673+58.
5. 765987÷94; 876392+106.
6. 5,843721÷290; 632,087631+43100.
7. 31,973256÷4720; 8,124970÷5673.
8. Divide 827579 by 108, and also successively by 9 and by 12, and show that the remainder and quotient are the same in each case.
9. Similarly divide 932854 by 63 and by 7 and 9.
10. Similarly divide 8,437985 by 132 and by 11 and 12.
11. Similarly divide 2,147952 by 168 and by 6, 4 and 7.
12. Similarly divide 85,437894 by 480 and by 10, 12 and 4.
13. Multiply 53729 by 229 and prove by division.
14. Multiply 87635 by 4008 and prove by division.
15. Multiply 96327 by 853 and prove by division.
16. Divide 7,864391 by 218 and prove by multiplication.
17. Divide 8,479732 by 497 and prove by multiplication.
18. Divide 10,573629 by 19300 and prove by multiplication.

The processes of addition, subtraction, multiplication, and division have now been sufficiently explained, and the pupil

may be supposed able to work out any examples of them. But there are some considerations connected with multiplication and division which naturally come in at this part of the subject, and which it is especially important to notice, because they point out the connection between these first elements of arithmetic and the higher parts of the subject to be considered presently.

Let the instance already mentioned of 59 divided by 8 be farther considered. The question was then put in the form 'How many times is 8 contained in 59?' and the answer was, 'It is contained 7 times, and there is besides a remainder 3 left over.' It is certain, then, that here, out of the given number 59, 56 has been divided by 8, and 3 has been left undivided. There is evidently in this something incomplete. Next suppose that the question were put in another form: 'What number would, when multiplied by 8, produce 59?' The answer to this cannot be more exact at present than 'A number greater than 7 and less than 8; and such a number we have no means of expressing.' The same incompleteness is found here, and it must be considered whether this arises from some actual impossibility in the question itself, or some imperfection in our mode of expressing results. This doubt is at once resolved by a reference to facts. It is evidently possible that 59 pounds of bread, or 59 gallons of ale, should be divided equally among eight people, and one share multiplied by eight, must be equal to the 59 pounds or 59 gallons. The difficulty, therefore, merely resolves itself into this, that we have, at present, no means of expressing the value of such a share; and consequently we must, if possible, extend our powers of expression. Now all the considerations that have been stated point to one conclusion, namely, that 59 divided by 8 is greater than 7 by a quantity that expresses the unknown result of dividing 3 by 8. We may therefore say that 59 divided by 8, is equal to 7 together with 3 divided by 8, or expressing the result with signs instead of words, $59 \div 8 =$

$7+(3\div 8)$; or, more shortly, $59\div 8=7+\frac{3}{8}$; or, the sign $+$ being left out, $59\div 8=7\frac{3}{8}$. Here it must be distinctly understood, that the expression $\frac{3}{8}$ is put for the result of a process that we have no simpler means of expressing, but which result has a certain definite value. Such an expression is called a *fraction*, the upper number being called the *numerator*, and the lower the *denominator*.

The fractions $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$, are read 'one-half,' 'one-fourth,' or 'one-quarter,' and 'three-fourths,' or 'three-quarters.' All other fractions may be read in two different ways, either by using the word 'by' to signify division, as when $\frac{39}{81}$, $\frac{53}{97}$, are called '39 by 81,' '53 by 97,' or by using the ordinal number expressed by the denominator, as when $\frac{39}{81}$, $\frac{53}{97}$, are called 'thirty-nine eighty-firsts,' 'fifty-three ninety-sevenths.' The latter form is almost exclusively used in the case of fractions with small denominators, as $\frac{3}{5}$ 'three-fifths,' $\frac{4}{7}$ 'four-sevenths,' and the reasons which led to its adoption will be explained in a future chapter.

Some knowledge of the nature and properties of fractions is absolutely necessary before the subject of Concrete Numbers can be considered, and what is sufficient for that purpose will be briefly explained here, postponing the full consideration of fractions to an after part of the book.

$\frac{35}{72}$ means the result of a division, which result cannot be expressed in our system of numbers. In this division 35 is the dividend, 72 is the divisor, and the value of the fraction $\frac{35}{72}$ is the quotient. And the laws that apply to dividends, divisors, and quotients, generally, must be the same whether the result is capable of expression as a whole number or not. Now, in the case of $\frac{84}{6}$, which we know is equal to 14, we find the following facts, from which, and similar instances, certain laws may be deduced.

$\frac{84}{6}=14$; $\frac{168}{6}=28$, therefore multiplying the dividend multiplies the quotient.

$\frac{84}{6}=14$; $\frac{84}{12}=7$, therefore multiplying the divisor divides the quotient.

$\frac{84}{6}=14$; $\frac{42}{6}=7$, therefore dividing the dividend divides the quotient.

$\frac{84}{6}=14$; $\frac{84}{3}=28$, therefore dividing the divisor multiplies the quotient.

Consequently $\frac{84}{6}\times 2=$ either $\frac{168}{6}$ or $\frac{84}{3}$;

$\frac{84}{6}\div 2=$ either $\frac{84}{12}$ or $\frac{42}{6}$.

Hence, in the case of $\frac{35}{72}$, the same laws must apply, and

$$\frac{35}{72}\times 6=\text{either } \frac{210}{72} \text{ or } \frac{35}{12},$$

$$\frac{35}{72}\div 5=\text{either } \frac{35}{360} \text{ or } \frac{7}{72}.$$

Expressing these rules generally, we have the following rules :—

To multiply a fraction : *Either multiply the numerator, or divide the denominator.*

To divide a fraction : *Either multiply the denominator, or divide the numerator.*

Consequently, if both the numerator and denominator of a fraction are multiplied or divided by the same number, the value of the fraction will be unaltered. Thus—

$$\frac{5}{8}=\frac{10}{16}=\frac{50}{80}=\frac{25}{40}=\frac{275}{440}=\frac{55}{88}.$$

It is evidently convenient, however, to express fractions in a form involving as low numbers as possible.

Since the order of factors in multiplication is immaterial, that is to say, since $8\times 11=11\times 8$, it follows that $\frac{12}{37}\times 5=5\times\frac{12}{37}$, and this latter must therefore be equal to $\frac{60}{37}$. Therefore 5, when multiplied by $\frac{12}{37}$, gives as product $\frac{60}{37}$, that is to say, the quantity which would result from multiplying 5 by 12, and dividing the product by 37. Hence to multiply by a fraction, the rule is, *Multiply by the numerator, and divide by the denominator.* Thus,

$$1780\times\frac{3}{5}=\frac{5340}{5}=1068.$$

As Multiplication and Division are contrary operations, it follows at once, from the above, that to divide by a fraction, the rule is, *Multiply by the denominator, and divide by the numerator.* Thus—

$$1068\div\frac{3}{5}=\frac{5340}{3}=1780.$$

It is sometimes required, in the Arithmetic of Concrete Numbers, to determine the quotient when a number, consisting of a whole number and a fraction, is divided by a whole number. As an example, let it be required to divide $8\frac{4}{5}$ by 12. Now $8\frac{4}{5}$ is the sum of $8+\frac{4}{5}$, and 8 is evidently equal to $\frac{40}{5}$. Hence $8\frac{4}{5}=\frac{40}{5}+\frac{4}{5}=\frac{44}{5}$, and $8\frac{4}{5}\div 12=\frac{44}{5}\div 12=\frac{44}{60}$, by the rule previously given for dividing a fraction. And the result, $\frac{44}{60}$, may be more shortly expressed as $\frac{11}{15}$. A number consisting of a whole number and a fraction, is called a *mixed number*, and the fraction to which it can be reduced must always have the numerator greater than the denominator. Such a fraction is termed an *improper fraction*. Mixed numbers, consequently, can be reduced to improper fractions, and improper fractions to mixed numbers. Thus—

$$4\tfrac{3}{7}=\tfrac{28}{7}+\tfrac{3}{7}=\tfrac{31}{7};\ 5\tfrac{2}{5}=\tfrac{25}{5}+\tfrac{2}{5}=\tfrac{27}{5};$$

or, the intermediate step being omitted, when the reason of the process is understood—

$$4\tfrac{3}{7}=\tfrac{31}{7};\ 5\tfrac{2}{5}=\tfrac{27}{5}.$$

Again—

$$\tfrac{56}{11}=5\tfrac{1}{11};\ \tfrac{23}{9}=2\tfrac{5}{9};\ \tfrac{84}{17}=4\tfrac{16}{17}.$$

Hence the general rule for dividing a mixed number by a whole number will be, *Reduce the mixed number to an improper fraction, and divide by the whole number.*

It is better, though not absolutely necessary, to express the fraction in the result with as small a numerator and denominator as possible.

$$8\tfrac{7}{11}\div 15=\tfrac{95}{11}\div 15=\tfrac{95}{165}=\tfrac{19}{33}.$$
$$7\tfrac{3}{8}\div 12=\tfrac{59}{8}\div 12=\tfrac{59}{96}.$$
$$11\tfrac{1}{5}\div 6=\tfrac{56}{5}\div 6=\tfrac{56}{30}=1\tfrac{26}{30}=1\tfrac{13}{15}.$$

Examples will now be given illustrating the various points that have been explained, namely—

Division of one number by another, the complete quotient being generally expressed as a whole number and a fraction, as $1937\div 32=60\frac{17}{32}$; $26853\div 48=559\frac{21}{48}=559\frac{7}{16}$.

Multiplication of a fraction or mixed number, as, $\frac{3}{7}\times 8=\frac{24}{7}=3\frac{3}{7}$; $\frac{43}{96}\times 12=\frac{43}{8}=5\frac{3}{8}$; $\frac{5}{28}\times 16=\frac{80}{28}=2\frac{24}{28}=2\frac{6}{7}$; $108\frac{7}{12}\times 14=1520\frac{2}{12}=1520\frac{1}{6}$.

Division of a fraction or mixed number, as, $\frac{14}{15} \div 7 = \frac{2}{15}$; $\frac{112}{225} \div 48 = \frac{12}{800} = \frac{7}{875}$; $3\frac{3}{7} \div 8 = \frac{24}{7} \div 8 = \frac{3}{7}$; $11\frac{5}{9} \div 16 = \frac{104}{9} \div 16 = \frac{104}{144} = \frac{13}{18}$.
Multiplication or division by a fraction, as, $20 \times \frac{7}{9} = \frac{140}{9} = 15\frac{5}{9}$; $13 \div \frac{5}{6}$ $\frac{78}{5} = 15\frac{3}{5}$.

Ex. 5.

1. $87632 \div 5$; $48936 \div 9$; $58374 \div 8$.
2. $9617 \div 11$; $34591 \div 14$; $92873 \div 31$.
3. $8643 \div 99$; $82696 \div 112$; $38725 \div 80$.
4. $\frac{5}{11} \times 2$; $\frac{13}{16} \times 9$; $\frac{15}{64} \times 8$; $\frac{17}{49} \times 7$.
5. $2\frac{7}{8} \times 12$; $4\frac{11}{15} \times 20$; $\frac{19}{30} \times 30$; $17\frac{2}{9} \times 24$.
6. $3\frac{1}{4} \times 108$; $17\frac{2}{5} \times 6$; $21\frac{25}{27} \times 36$; $43\frac{18}{25} \times 30$.
7. $\frac{32}{45} \div 8$; $\frac{171}{200} \div 19$; $\frac{35}{48} \div 28$; $\frac{4}{9} \div 20$.
8. $\frac{19}{25} \div 3$; $\frac{29}{42} \div 7$; $6\frac{63}{80} \div 12$; $9\frac{27}{56} \div 42$.
9. $11\frac{2}{3} \div 7$; $14\frac{3}{10} \div 11$; $8\frac{5}{9} \div 21$; $10\frac{2}{11} \div 63$.
10. $62\frac{1}{2} \div 100$; $3\frac{3}{4} \div 10$; $17\frac{1}{3} \div 100$; $187\frac{1}{2} \div 1000$.
11. $12 \times \frac{5}{8}$; $24 \times \frac{3}{4}$; $100 \times \frac{4}{5}$; $96 \times \frac{13}{15}$.
12. $24 \times \frac{6}{7}$; $112 \times \frac{7}{8}$; $90 \div \frac{3}{7}$; $24 \div \frac{18}{25}$.
13. $48 \div \frac{100}{111}$; $75 \div \frac{60}{89}$; $81 \div \frac{30}{41}$; $100 \div \frac{65}{72}$.

CHAPTER II.

CONCRETE NUMBERS.

A CONCRETE NUMBER has been defined to oe a number of things of some particular kind, as three yards, four hours, seven mules, &c.

A quantity is frequently expressed by several concrete numbers of different denominations, as, for instance, a certain weight may be 3 tons 9 cwt. 2 qrs. 5 lb.

The relations which subsist between the different denominations of the same kind are to be found in the tables which are now given.

ARITHMETICAL TABLES.

I. TIME.

This measure depends upon the length of the mean solar day.

60 seconds make	1 minute.
60 minutes „	1 hour.
24 hours „	1 day.
7 days „	1 week.
$52\frac{1}{7}$ weeks or 365 days . . .	1 year.

The year is also divided into 12 *Calendar Months* of unequal length, April, June, and November containing 30 days, February having 28, and all the rest 31. Every fourth year is called a *Leap Year*, and contains 366 days, February having 29 instead of 28.

4 weeks make 1 *Lunar Month.*

II. LENGTH.

This measure depends upon the length of a pendulum vibrating seconds in the latitude of London. The standard is a *yard*, which is equal to ·919792 of this length.

12 inches make	1 foot.
3 feet „	1 yard.
$5\frac{1}{2}$ yards „	1 pole.
40 poles „	1 furlong.
8 furlongs „	1 mile.
3 miles „	1 league.

For cloth measure a separate table is used, where—

$2\frac{1}{4}$ inches make	. . . .	1 nail.
4 nails ,,	. . . .	1 quarter.
4 quarters ,,	. . . .	1 yard.
5 quarters ,,	. . . .	1 ell.

In land surveying 100 *links* make 1 *chain*, the chain being 66 feet, and a link consequently being $7\frac{23}{25}$ inches. 80 chains are equal to 1 mile.

Other less important measures are—

3 *barleycorns* make	. . .	1 inch
4 inches ,,	. . .	1 *hand*
6 feet ,,	. . .	1 *fathom.*

The length of the circumference of any circle is supposed to be divided into 360 equal parts. Each of these is called a *degree*, the degree is divided into 60 *minutes*, and the minute into 60 *seconds*. The abbreviations for these three words are $^{\circ}\ '\ ''$, so that 5 degrees 27 minutes 34 seconds is written $5^{\circ}\ 27'\ 34''$. The length of a degree of any circle depends entirely upon the size of the circle. In the particular case of the earth's circumference—

A degree of the equator = 69·1639 miles,
A mean degree of the meridian = 69·0488 miles,

or nearly $69\frac{1}{6}$ and $69\frac{1}{20}$ miles respectively.

The degrees of the meridian vary from 68·702 at the equator to 69·396 at the pole.

A minute of the equator is also called a *geographical* or *nautical* mile.

The table of measures of length leads directly to those of measures of surface and solidity, by squaring and cubing the numbers contained in it respectively.

Thus 144 square inches make		1 square foot.
9 square feet	,,	1 square yard.
$30\frac{1}{4}$ square yards	,,	1 square pole (or 1 *perch*).

At this point the table of square measure ceases to have an analogy to the table of length, but continues—

40 perches make	. . . .	1 rood.
4 roods ,,	. . . .	1 acre.

The acre contains 4840 square yards, or 10 square chains, and 640 acres make 1 square mile.

In solid or cubic measure

1728 cubic inches make	. .	1 cubic foot.
27 cubic feet ,,	. .	1 cubic yard.

III. CAPACITY.

The standard is the *gallon*, which contains 277·274 cubic inches.

4 gills	make	. . . . .	1 pint.
2 pints	„	. . . . .	1 quart.
4 quarts	„	. . . . .	1 gallon.
2 gallons	„	. . . . .	1 peck.
4 pecks	„	. . . . .	1 bushel.
8 bushels	„	. . . . .	1 quarter.
5 quarters	„	. . . . .	1 load.

A barrel of beer contains 36 gallons, and a hogshead 54 gallons.

IV. WEIGHT.

The standard is the *pound avoirdupois*, which is the weight of one tenth of a gallon of distilled water, when the barometer is at 30° and the thermometer at 62°. The pound weight actually constructed as a standard, is a cylinder of platinum nearly 1·35 inches high and 1·15 inches in diameter.

16 drams	make	. . . .	1 ounce (oz.)
16 ounces	„	. . . .	1 pound (lb.)
28 pounds	„	. . . .	1 quarter.
4 quarters	„	. . . .	1 hundredweight (cwt.)
20 hundredweight		. . . .	1 ton.

The *stone* is 14 lb.

For weighing gold, silver, jewels, and in philosophical experiments, *troy weight* is used, the grain troy being the seven thousandth part of the standard pound avoirdupois.

24 grains	make	. .	1 pennyweight (dwt.)
20 pennyweights	make	.	1 ounce.
12 ounces	„	.	1 pound.

In compounding medicine, the troy grain, ounce, and pound are used, but the ounce is differently subdivided; thus,

20 grains	make	. . . .	1 scruple.
3 scruples	„	. . . .	1 drachm.
8 drachms	„	. . . .	1 ounce.

A cubic foot of water weighs nearly 1000 ounces avoirdupois. This being easily remembered, and very nearly correct, will be taken as the weight of water in all examples in this book, unless the contrary be expressly stated.

V. VALUE.

The standard is a gold coin called a *pound sterling* or a *sovereign*, weighing 123·274 grains, and made of a metal containing 22 parts of pure gold and 2 parts of copper.

A *shilling* is defined to be a coin weighing 3 dwts. $15\frac{3}{11}$ grains, or $\frac{1}{66}$ of a lb. troy, and made of a metal composed of 37 parts of pure silver and 3 parts of copper.

A *penny* used formerly to be a copper coin weighing $\frac{1}{24}$ of a lb. avoirdupois, but lately a bronze coinage has been substituted.

The shilling and penny have a fixed value, determined by authority, such that

12 pence make . . .	1 shilling.
20 shillings " . . .	1 pound.

The abbreviations for the words pounds, shillings and pence, are *£ s. d.*

In order to prevent silver and copper (or bronze) coins from acquiring a different value from that fixed by law, it is enacted that silver is not a legal tender for more than 40 shillings, nor copper (or bronze) for more than 12 pence.

The gold coins now in use are the sovereign and the half sovereign.

The silver coins are the crown (5*s.*); the half-crown (2*s.* 6*d.*); the florin (2*s.*); the shilling, sixpence, fourpence, and threepence.

The copper (or bronze) coins are the penny, halfpenny, and farthing ($\frac{1}{4}$*d.*)

A *guinea*, a gold coin formerly in circulation, was worth 21*s.*

In each of these tables some one denomination must be accurately defined, and the others may then be found from it. The following is a brief explanation of this point.

All our English weights and measures depend ultimately upon the sun's apparent motion in the heavens, and its effect in dividing time into days and years. From the length of the 'mean solar day' we obtain that of an hour, a minute, and a SECOND. From this we can ascertain the length of a pendulum that will vibrate seconds exactly. The YARD is defined to be in a certain proportion to this, and thus a standard measure of length is obtained. Having deduced from this inches, feet, miles, &c., square inches, square feet, &c., cubic inches, cubic feet, &c., we obtain a standard measure of capacity, the GALLON, which is defined to be such as to contain a certain number of cubic inches. Next, the weight of a gallon of water at a certain temperature is fixed by authority as the weight of ten pounds avoirdupois, and thus the POUND, the standard measure of weight, is

obtained. From one pound weight of a metal composed of 22 parts of gold to 2 parts of copper, a certain number of coins are made. One of these is called a SOVEREIGN, and this is the standard measure of value.

Practically, it is found convenient to have a yard measure and a pound weight accurately constructed, and made of durable material, authorised by Parliament, and placed in the charge of a public officer. Should it happen, however, that these were lost or damaged, they could be restored by reference to the above definitions, although care has been taken to prevent such a necessity by having several copies of each constructed, and kept in different places. A description of these standards, and a variety of curious and useful information on the subject of weights and measures, may be found in any number of Dietrichsen & Hannay's Royal Almanack.

When a quantity is expressed in several denominations, it may be reduced to one the lowest of these, by multiplying in succession by the numbers necessary to bring each denomination to the next lower, and adding in whatever of that lower denomination may have been given in the original expression. For example:

Reduce 18 cwt. 3 qr. 11 lb. 4 oz. to drams;
„ 8 years 13 days 6 hours 21 sec. to seconds.

cwt. qr. lb. oz.		yrs. dys. hr. sec.	
18 3 11 4		8 13 6 21	
4		365	
75		2933	
28		24	
611		11738	
150	540480 drams.	5866	253432821 seconds.
2111		70398	
16		60	
12670		4223880	
2111		60	
33780		253432821	
16			
202680			
33780			
540480			

When a quantity is expressed in one denomination, it may be brought to several denominations by dividing in succession by the numbers necessary to bring each denomination to the next higher, and the successive remainders and last quotient will be the several denominations required. To illustrate this, the preceding examples may be worked back again, the questions standing thus :

Reduce 540480 drams to cwt.; 253432821 seconds to years.

8)540480	60)25343282,1
2) 67560	60) 422388,0 − 21
8) 33780	4) 70398 − 0
2) 4222 − 4 } 4	6) 17599 − 2 } 6
7) 2111 − 0	2933 − 1
4) 301 − 4 } 11	365)2933(8
4) 75 − 1	2920
18 − 3	13
18 cwt. 3 qr. 11 lb. 4 oz.	8 yrs. 13 dys. 6 hr. 0 m. 21 sec.

These processes are both called *Reduction*, and before any more examples are given the following observations may be made.

I. The system of weights and measures used in England is extremely cumbrous and inconvenient. This is owing to the variety of numbers used to express the relations between different denominations. If a new system were to be established, there is no doubt that the most convenient one would be where each denomination was one-tenth part of that next higher. Concrete numbers would then follow the same law of local value as abstract numbers, and all processes and operations would be the same in each case. By this means fully one-half of the difficulties and complexities of arithmetic would be avoided. The inconvenience, however, of changing what is now firmly established, has hitherto prevented anything being done to render our present

system more simple. As an instance of the advantages of the 'decimal system,' as it is termed, we will suppose that the table of money has been altered to the following :—Ten mils = one cent; ten cents = one florin; ten florins = £1. The amount £137 *5fl.* *6c.* *7m.* would then be more simply written 137567*m.*; £25 *8c.* would be written 25080*m.*; £100 would be 100,000*m.*; *7fl.* *6c.* *5m.* would be 765*m.* Let it be required to work out the following questions :—

(1.) Add together the amounts just given, and from their sum subtract 258732 m.

(2.) Divide 917125 m. among 29 persons.

(3.) Relief is given to the people of a town in a time of scarcity to the following extent: to 1273 families of six, 1 fl. each per week; to 475 of five, 1 fl. 2 c.; to 372 of four, 1 fl. 4 c.; to 367 of three, 1 fl. 6 c.; to 279 of two, 1 fl. 8 c.; and to 457 single persons, 2 fl. each. What is the total amount expended in 11 weeks?

(1.)

```
137567
 25080
100000
   765
------
263412
258732
------
  4680
```

or £4 *6fl.* *8c.*

(2.)

```
29)917125(31625
   87
   --
    47
    29
    --
    181
    174
    ---
      72
      58
      --
      145
      145
      ---
```

or £31 *6fl.* *2c.* *5m.*

(3.)

```
  1273        475         372
   600        600         560
------     ------      ------
763800     285000       22320
                        1860
                      -------
                       208320
  367         279      763800
  480         360      285000
------     ------      176160
 29360      16740      100440
 1468       837         91400
------     ------     -------
176160     100440     1625120
                           11
  457                 -------
  200                17876320
------
 91400
```

(Note D.) Result, £17876 *3fl.* *2c.*

II. Since $5\frac{1}{2}$ yards = 1 pole, $30\frac{1}{4}$ square yards = 1 perch, and $2\frac{1}{4}$ inches = 1 nail, it is best, when possible, to avoid these troublesome numbers, and to pass over them, making 220 yards = 1 furlong; 1210 square yards = 1 rood; and 9 inches = 1 quarter. As, however, it is sometimes requisite that these numbers should be used, the following example is given to show how the number $5\frac{1}{2}$ in $5\frac{1}{2}$ yards = 1 pole is to be managed :—

Reduce 18 m. 3 fur. 33 p. 2 yd. 2 ft. 7 in. to inches, and verify the result by reducing back to miles.

```
 m.  f.  p.  yd. ft. in.
 18  3   33  2   2   7
  8
147
 40
5913      1170877 inches.
   5½
29567
 2956½
32523½
    3
97572½
   12
1170877
```

```
⎧ 6)1170877
⎩ 3) 195146— 1 ⎫
 11)  65048— 2 ⎭ 13 inches.
 40)   591,3— 5 half yards.
  8)   147 —33
        18 — 3

 m.  f.  p.  yd. ft. in.
 18  3   33  2   1   1
                 1   6
 18  3   33  2   2   7
```

III. In all examples where time is involved, it is especially necessary that the question should be exactly stated. A year is generally considered to be 365 days, more exactly it would be taken as $365\frac{1}{4}$ days, and even this is not perfectly accurate. Sometimes, however, a year is considered as being 12 months; and if there are 30 days in each, the year would be 360 days. Again, sometimes a year is considered as 52 weeks of 7 days each, or 364 days. Consequently, particular care should be taken that the question should be so definitely stated as to leave no room for ambiguity. Throughout this book, unless the contrary is expressly stated, the year is always intended to mean 365 days.

IV. In the Table of Money there is in one part a different mode of expression from that used in all other tables. The lowest denomination is farthings, and 4 farthings = 1 penny.

Now, farthings are never, or at any rate very seldom, expressed as whole numbers, but always as fractions of a penny. Thus one farthing, or one-fourth of a penny, is written $\frac{1}{4}$; two farthings, or one-fourth of two pence, is written $\frac{1}{2}$, because $\frac{1}{2} = \frac{2}{4}$; three farthings, or one-fourth of three pence, is written $\frac{3}{4}$. In reduction, addition, subtraction and multiplication of money it is best to consider these expressions, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, as peculiar modes of signifying one farthing, two farthings, three farthings, &c. In division of money, it is best to consider them as meaning certain fractions of a penny, because, in any quotient resulting from such a division, the part less than one penny is never, or at any rate very seldom, expressed as farthings and a fraction of a farthing, but as a fraction of a penny (Note E.) This point will be illustrated by examples presently.

V. Sometimes the reduction from one denomination to another cannot be effected directly, but some third denomination must be used as a medium between them. Thus, to reduce 783621 fourpences to half-crowns, it would be necessary to reduce them first to pence by multiplying by 4, and then to reduce the pence to half-crowns by dividing by 30.

Before giving examples to be worked out by the pupil, a few are now given illustrating some of the points where difficulties may occur.

```
   783621  fourpences to
        4     half-crowns.
30)313448.4
   104482—24
Ans. 104482 half-crowns 2s.
```

```
  86931  half-guineas 4s. 6d. to
     21     half-crowns.
  86940  (adding in the nine six-
 173862     pences in 4s. 6d.)
5)1825560
  365112 half-crowns. Ans.
```

```
    13 lb. av. to troy weight.
     7000
 4)  91000
 6)  22750 – 0 ┐
20)   3791 – 4 ┘ 16
12)    189 – 1
        15 – 9
Ans. 15 lb. 9 oz. 1 dwt. 16 gr.
```

```
  7  drachms. ap. weight to troy
 60     weight.
4)420
6)105 – 0 ┐
   17 – 3 ┘ 12
Ans. 17 dwt. 12 gr.
```

$$
\begin{array}{rl}
12)\underline{825745} & \text{square inches to perches.} \\
3)\underline{\;68812-1} & \left.\right\} \text{13 inches.} \\
9)\underline{\;22937-1} & \\
11)\underline{\;\;2548-5} & \text{quarter feet.} \\
11)\underline{\;\;\;231-7} & \left.\right\} \text{7 quarter yards.} \\
21-0 &
\end{array}
$$

perches	yd.	ft.	in.
21	1	6	108
		1	36
			13
21	1	8	13. *Ans.*

Ex. 6.

1. £2743 3*s*. 5*d*. to pence, and 972862 farthings to pounds.
2. 1648 crowns 3*s*. 5*d*. to farthings, and 30783 half-crowns and 6*d*. to shillings.
3. 9368540 threepences to pounds, and £4939 13*s*. 8*d*. to fourpences.
4. 347 guineas 15*s*. 1*d*. to pence, and £174 13*s*. 7½*d*. to halfpence.
5. 378521 fourpences to half-guineas, and 189557 half-guineas 6*s*. 6*d*. to half-crowns.
6. 83561 threepences to crowns, and £50543 6*s*. 6*d*. to half-guineas.
7. 81795 pence to pounds, and 894715 farthings to pounds.
8. 458631 sixpences to guineas, and 963245 farthings to florins.
9. 24 cwt. 1 qr. 1 oz. 15 dr. to drams, and 20 tons 15 cwt. 1 oz. to ounces.
10. 9854321 ounces to quarters, and 85726943 drams to tons.
11. 98032 cwt. 1 qr. 1 stone to stones, and 635297 drams to stones.
12. 136 lb troy. 4 oz. 3 dwt. 9 gr. to grains, and 968439 grains to lbs. troy.
13. 127 dwt. to apothecaries' weight, and 28 lb. avoirdupois to troy weight.
14. 70 loads 1 qr. 6 bus. 3 pks. 1 gal. to gills, and 156495 pints to loads.
15. 956765 gal. 1 qt. 1 pt. to pints, and 69851 quarts to bushels.
16. 119 miles 5 fur. 91 yd. 1 ft. 6 in. to inches, and 986547 inches to miles.
17. 320 poles 5 yd. 9 in. to inches, and 106701 inches to poles.
18. 127 miles 4 fur. 38 poles 2 yd. to feet, and 876325 feet to leagues.
19. 535 sq. yd. 1 ft. 39 in. to square inches, and 762841 square inches to square yards.

20. 14 ac. 1 rd. 450 yd. 3 ft. to square feet, and 376294 square yards to roods.

21. 986427 square inches to perches, and 5,000000 square yards to square miles.

22. 185 cub. yd. 9 ft. 323 in. to cubic inches, and 9287631 cubic inches to cubic yards.

23. 3 years 45 days 13 hrs. 13 min. 32 sec. to seconds, and 7496251 minutes to years.

24. 154 weeks 1 day 14 hrs. 4 min. 5 sec. to seconds, and 89386 minutes to days.

The addition, subtraction, multiplication, and division of concrete numbers are generally termed *compound addition, compound subtraction*, &c.

To add together concrete numbers of several denominations, begin with those of the lowest, and reduce the sum of each denomination to the next higher, putting down the remainder, if any, and carrying forward the number of the next higher denomination to its own column.

Similarly in subtraction, subtract the lower number of each denomination beginning with the lowest from the upper, and when necessary borrow one from the next higher denomination, afterwards adding it on to the lower number in that denomination.

It will be seen that the only difference between these and the addition and subtraction of ordinary numbers is, that in the one case there is a constantly changing scale of local value, and in the other case an unchanging scale of 10 increasing towards the left.

EXAMPLES.

£	*s.*	*d.*
18	7	2
25	13	$6\frac{1}{2}$
54	17	$4\frac{3}{4}$
28	11	9
15	13	$7\frac{1}{2}$
143	3	$5\frac{3}{4}$

tons	cwt.	qr.	lb.	oz.	dr.
11	3	2	14	14	9
17	13	1	19	5	15
8	11	3	25	14	10
23	17	1	23	11	12
6	11	0	17	8	7
15	10	3	21	9	14
24	5	1	20	11	6
107	14	0	3	12	9

Subtraction.

lbs. troy.	oz.	dwt.	gr.
13	6	13	2
8	4	15	7
5	1	17	19

bush.	pk.	gal.	qt.	pt.
17	1	0	3	1
11	3	1	2	0
15	1	1	1	1

As in reduction, it is better, if possible, to avoid the denominations of poles, perches, and nails. When this cannot be done, the division by $5\frac{1}{2}$ to reduce yards to poles will frequently leave as remainder some number of yards and a half. This half-yard should be added to the feet and inches, as was done in reduction.

EXAMPLES.

Subtraction.

	poles	yd.	ft.	in.
	13	3	1	4
	7	3	1	9
	5	$4\frac{1}{2}$	2	7
or,	poles	yd.	ft.	in.
	5	5	1	1

Addition.

	roods	perch.	yd.	ft.	in.
	3	17	$16\frac{1}{4}$	3	41
	2	30	$7\frac{1}{2}$	5	120
	6	7	$24\frac{3}{4}$	0	17
or,	roods	perch.	yd.	ft.	in.
	6	7	24	6	125

It may be noticed that, owing to the complexity of these portions of the tables, the same result may appear in different forms. Thus 3 poles 5 yd. 2 ft. 8 in. and 4 poles 0 yd. 1 ft. 2 in. mean the same length; 7 perches 30 yd. 7 ft. 87 in. and 8 perches 0 yd. 5 ft. 51 in. mean the same area. In either case the latter is the preferable form.

Ex. 7.

Add together:—

1. £10 5*s*. $3\frac{1}{4}$*d*., £18 19*s*. $2\frac{1}{2}$*d*., £34 7*s*. 5*d*., £5 0*s*. $11\frac{3}{4}$*d*., and £16 13*s*. $2\frac{1}{2}$*d*.

2. £18 17*s*. $6\frac{3}{4}$*d*., £21 15*s*. $10\frac{1}{2}$*d*., £13 17*s*. 5*d*., £42 11*s*. $8\frac{1}{2}$*d*., and £9 19*s*. 10*d*.

3. £13 15*s*. 9*d*., £36 11*s*. $2\frac{1}{2}$*d*., £17 19*s*. $4\frac{3}{4}$*d*., £32 8*s*. 7*d*., and £84 11*s*. $6\frac{1}{4}$*d*.

4. £6 14*s*. $2\frac{1}{2}$*d*., £7 11*s*. 3*d*., £9 4*s*. $2\frac{3}{4}$*d*., £5 19*s*. $10\frac{1}{4}$*d*., and £2 17*s*. $6\frac{1}{2}$*d*.

5. 11 tons 14 cwt. 2 qr. 9 lb., 3 tons 15 cwt. 1 qr. 23 lb., 10 tons 7 cwt. 3 qr. 19 lb., 5 tons 13 cwt. 24 lb., and 4 tons 2 cwt. 1 qr. 6 lb.

6. 3 qr. 12 lb. 11 oz. 9 dr., 2 qr. 21 lb. 15 oz. 4 dr., 3 lb. 11 oz., 1 qr. 19 lb. 14 oz. 8 dr., and 2 qr. 7 oz.

7. 5 lb. 3 oz. 17 dwt. 8 gr., 2 lb. 9 oz. 19 dwt. 10 gr., 7 lb. 8 oz. 4 dwt. 22 gr., 6 lb. 5 oz. 18 dwt. 11 gr., and 3 lb. 2 oz. 17 dwt. 4 gr.

8. 13 bus. 1 pk. 3 qt., 17 bus. 1 gal. 3 pk. 2 qt., 11 bus. 2 pk. 1 qt., and 5 bus. 1 gal. 3 qt.

9. 3 ld. 2 qr. 5 bus. 6 gal., 4 ld. 3 qr. 2 bus. 5 gal., 3 ld. 4 qr. 7 bus. 2 gal., 6 ld. 1 qr. 5 bus. 3 gal., and 7 ld. 6 bus.

10. 5 lea. 1 mi. 6 fur. 29 po., 3 lea. 2 mi. 7 fur. 33 po., 6 lea. 1 mi. 5 fur. 18 po., 4 lea. 3 fur. 20 po., and 7 lea. 1 mi. 6 fur. 9 po.

11. 17 mi. 2 fur. 119 yd. 1 ft., 23 mi. 5 fur. 37 yd. 2 ft., 35 mi. 3 fur. 218 yd., 47 mi. 6 fur. 120 yd. 1 ft., and 9 mi. 5 fur. 13 yd.

12. 3 po. 2 yd. 1 ft. 9 in., 5 po. 4 yd. 2 ft. 8 in., 6 po. 3 yd. 7 in., 4 po. 4 yd. 2 ft. 10 in., and 2 po. 1 yd. 6 in.

13. 19 po. 2 yd. 1 ft. 11 in., 13 po. 1 yd. 8 in., 6 po. 2 ft. 9 in., 10 po. 1 yd. 1 ft. 4 in., and 3 po. 3 yd. 1 ft. 7 in.

14. 2 ells 3 qr. 2 n., 4 ells 1 qr. 1 n., 5 ells 3 n., and 4 ells 2 qr.

15. 31 sq. yd. 8 ft. 109 in., 14 yd. 7 ft. 54 in., 12 yd. 6 ft. 43 in., 8 yd. 5 ft. 111 in., and 19 yd. 7 ft. 86 in.

16. 13 ac. 2 rd. 597 yd., 25 ac. 1 rd. 153 yd., 9 ac. 317 yd., 41 ac. 2 rd. 1125 yd., and 36 ac. 1 rd. 809 yd.

17. 5 ac. 1 rd. 23 per. 11 yd., 6 ac. 2 rd. 9 per. 25 yd., 4 ac. 13 per. 8 yd., 2 ac. 3 rd. 25 per. 23 yd., and 3 ac. 2 rd. 7 per. 28 yd.

18. 37 cub. yd. 23 ft. 894 in., 51 yd. 19 ft. 435 in., 13 yd. 7 ft. 1690 in., 42 yd. 15 ft. 311 in., and 29 yd. 3 ft. 1512 in.

19. 5 yr. 71 d. 13 hr. 29 m., 3 yr. 115 d. 2 hr. 47 m., 2 yr. 296 d. 19 hr. 53 m., 1 yr. 74 d. 15 hr. 42 m., and 4 yr. 186 d. 5 hr. 33 m.

20. 5 wk. 3 d. 13 hr., 2 wk. 4 d. 7 h., 6 wk. 5 d. 18 hr., 4 wk. 6 d. 9 hr., and 2 wk. 2 d. 10 hr.

Subtract:—

21. £17 9*s*. 2$\frac{3}{4}$*d*. from £43 11*s*. 9$\frac{1}{2}$*d*.

22. £29 10*s*. 10$\frac{1}{2}$*d*. from £58 19*s*. 6*d*.

23. 19 tons 18 cwt. 3 qr. 13 lb. from 23 tons 15 cwt. 2 qr. 7 lb.

24. 1 qr. 19 lb. 8 oz. 12 dr. from 2 qr. 13 lb. 15 oz. 11 dr.

25. 2 lb. 8 oz. 10 dwt. 7 gr. from 11 lb. 5 oz. 3 dwt. 19 gr.

26. 3 oz. 5 dr. 2 sc. 12 gr. from 5 oz. 3 dr. 2 sc. 7 gr.

27. 3 bus. 3 pk. 2 qt. from 5 bus. 2 pk. 1 gal. 3 qt.

28. 5 mi. 7 fur. 200 yd. 2 ft. from 7 mi. 6 fur. 112 yd. 1 ft.

29. 4 po. 5 yd. 2 ft. 7 in. from 7 po. 3 yd. 1 ft. 9 in.

30. 17 sq. yd. 5 ft. 96 in. from 24 sq. yd. 4 ft. 13 in.

31. 39 ac. 3 rd. 48 per. from 53 ac. 2 rd. 27 per.

32. 12 per. 4 yd. 3 ft. 41 in. from 16 per. 3 yd. 7 ft. 93 in.
33. 25 cub. yd. 17 ft. 809 in. from 36 cub. yd. 13 ft. 712 in.
34. 10 yr. 233 d. 6 hr. from 17 yr. 45 d. 11 h.

Subtract and prove by addition:—

35. £9 6*s.* $7\frac{1}{2}$*d.* from £15 13*s.* $2\frac{3}{4}$*d.*
36. £16 14*s.* $10\frac{3}{4}$*d.* from £21 11*s.* $7\frac{1}{2}$*d.*
37. £33 15*s.* $3\frac{3}{4}$*d.* from £47 8*s.* $9\frac{1}{2}$*d.*
38. £39 2*s.* $4\frac{1}{2}$*d.* from £71 11*s.* $10\frac{1}{4}$*d.*
39. 12 tons 11 cwt. 1 qr. 26 lb. from 15 tons 3 cwt. 2 qr. 19 lb.
40. 8 cwt. 2 qr. 19 lb. 11 oz. from 11 cwt. 1 qr. 15 lb. 2 oz.

In compound multiplication, multiply each denomination in succession, beginning with the lowest, and reduce the product to the next higher denomination, carrying on the number thus obtained, and setting down the remainder in its own column.

Multiplication of this kind differs from that of abstract numbers only in the fact of there being a varying instead of a constant scale of local value. The practical effect of this difference is to render multiplication by numbers greater than 12 much more troublesome than multiplication by numbers less than 12, and therefore each larger multiplier should be replaced by several smaller. Sometimes this may be done by resolving the multiplier into its factors. For example, let the multipliers be $495 = 5 \times 9 \times 11$ and $448 = 8 \times 8 \times 7$.

miles	fur.	yd.	ft.
7	3	82	1
			5
37	0	191	2
			9
333	7	185	0
			11
3673	6	55	0

acres	roods	perches
3	1	17
		8
26	3	16
		8
214	3	8
		7
1503	2	16

If the number is not one that can be separated into small factors, but is a little greater or less than one that can be so separated, this latter may be taken as the multiplier, and the difference multiplied into the multiplicand added to or

subtracted from the product. Thus, in the cases of $371 = (11 \times 11 \times 3) + 8$ and $527 = (11 \times 12 \times 4) - 1$.

tons	cwt.	qr.	lb.	oz.	
		2	8	15	× 8
				11	
	5	3	15	5	
				11	
3	4	3	0	7	
				3	
9	14	1	1	5	
	4	1	8	8	
9	18	2	4	13	

£	s.	d.	
1	15	$3\frac{1}{2}$	× 1
		11	
19	8	$2\frac{1}{2}$	
		12	
232	18	6	
		4	
931	14	0	
1	15	$3\frac{1}{2}$	
929	18	$8\frac{1}{2}$	

Supposing that 12 is the largest number to be used as a multiplier at one operation, the method just stated will be generally sufficient. In fact, all numbers, as far as 1420 inclusive, may be replaced by a set of multipliers each not greater than 12, arranged as in the examples given, that is to say, all but one being used as continued multipliers, and the product of the last and the multiplicand being either added or subtracted at the end. And, by proper arrangement of multipliers, each not greater than 12, multiplication by any number whatever may be effected in a somewhat similar manner. The trouble and difficulty of making these arrangements of multipliers may, however, be avoided by using a method which, though generally longer, is much more simple. For example, let it be required to multiply by 1427. Multiply by 10, and 10, and 10, this will give three lines of figures, equal to 10 times, 100 times, and 1000 times the multiplicand respectively. The sum, therefore, of (1 × third line) + (4 × second) + (2 × first) + (7 × multiplicand) will be the product required. As an example, multiply 7 dwt. 2 gr. by 1427. This is merely given as an illustration of the method just explained. It would be a shorter and easier plan to reduce the 7 dwt. 2 gr. to grains, and after multiplying by 1427, reduce to dwts., ounces, and pounds. The question is here worked out in both ways.

lb.	oz.	dwt.	gr.	
		7	2	× 7
			10	
	3	10	20	× 2
			10	
2	11	8	8	× 4
			10	
29	6	3	8	× 1
11	9	13	8	
	7	1	16	
	2	9	14	
42	1	7	22	

dwt.	gr.
7	2
24	
170	

$$\begin{array}{r}
1427 \\
\underline{170} \\
99890 \\
\underline{1427} \\
\left\{\begin{array}{r} 12)\underline{242590} \\ 2)\ \underline{20215} \end{array}\right. - 10 \\
20)\ \underline{1010,7} - \ \ 1 \\
12)\ \underline{505} - \ \ 7 \\
42 - \ \ 1
\end{array}
\left.\begin{array}{l} \\ \\ \\ \\ \\ \\ \end{array}\right\} 22$$

lb.	oz.	dwt.	gr.
42	1	7	22

n all cases judgment must be exercised, whether to uce to one denomination or not before multiplying. It l be seen, further on in this book, that the above question, well as most of the questions in Compound Multiplication, y be still more shortly worked out by a process called ractice.'

Ex. 8.

Find the values of:—

£3 18*s.* $3\frac{1}{4}$*d.* × 6.	2. £5 17*s.* $9\frac{1}{2}$*d.* × 9.
£11 13*s.* $8\frac{3}{4}$*d.* × 7.	4. £15 2*s.* $1\frac{1}{2}$*d.* × 11.
£23 7*s.* $7\frac{1}{4}$*d.* × 12.	6. 13 cwt. 1 qr. 11 lb. 9 oz. × 5.
6 lb. 4 oz. 11 dwt. 19 gr. × 4.	8. 16 mi. 7 fur. 151 yd. 2 ft. × 8.
3 po. 2 yd. 1 ft. × 4.	10. £17 13*s.* $4\frac{1}{2}$*d.* × 25.
£39 2*s.* $11\frac{3}{4}$*d.* × 28.	12. 3 ld. 1 qr. 5 bus. × 63.
4 yr. 153 d. 6 h. 19 m. × 84.	14. £3 19*s.* $5\frac{3}{4}$*d.* × 93.
£2 17*s.* $8\frac{1}{2}$*d.* × 125.	16. 3 qr. 11 lb. 15 oz. × 69
3 sq. yd. 7 ft. 92 in. × 83.	18. 5 ac. 2 rd. 26 per. × 1080.
£0 15*s.* $3\frac{1}{4}$*d.* × 328.	20. 17 lb. av. 11 oz. × 1137.

Prove that:—

£33 16*s.* 6*d.* × 91 = £36 12*s.* $10\frac{1}{2}$*d.* × 84.
£3 3*s.* $4\frac{1}{2}$*d.* × 1100 = £5 19*s.* 2*d.* × 585.
2 qr. 5 oz. × 345 = 1 qr. 21 lb. 11 oz × 391.
2 mi. 5 fur. 73 yd. × 713 = 3 mi. 1 fur. 181 yd. × 589.
1 yr. 210 d. 3 hr. × 629 = 1 yr. 129 d. 21 hr. × 731.
199 yd. 2 ft. 6 in. × 689 = 1 fur. 2 po. 5 yd. 6 in. × 585.
£3 15*s.* 3*d.* × 2527 = £5 19*s.* $1\frac{3}{4}$*d.* × 1596.
8 hr. 57 m. 51 sec. × 2419 = 11 hr. 51 m. 21 sec. × 1829.

Compound Division, by a divisor not greater than 12, y be performed at one operation, by dividing successively

each denomination, beginning with the highest, setting down the quotient, reducing the remainder to the next lower denomination, and carrying forward the number so obtained.

In division of money it will often occur that there are farthings, or fractions of a penny. In the cases of other concrete quantities also, there may be a fraction of the lowest denomination in the dividend. The method of treating such fractions has already been explained. (See p. 22.)

	£	s.	d.
6)	18	17	$9\frac{3}{4}$
	3	2	$11\frac{5}{8}$

	tons	cwt.	qr.	lb.	oz.
8)	18	3	3	7	
	2	5	1	25	6

If the divisor be greater than 12, but can be separated into factors each not greater than 12, then the division can be performed by means of two or more short divisions of the kind just given.

	£	s.	d.	
3)	26	14	$9\frac{1}{2}$	$\div$ 18
6)	8	18	$3\frac{1}{6}$	
	1	9	$8\frac{19}{36}$	

	miles	fur.	yd.	ft.	in.	
10)	7	2	25			$\div$ 480
6)		5	178	1	6	
8)			213	0	3	
			26	1	$10\frac{7}{8}$	

If the divisor be greater than 12, but cannot be separated into small factors, the method is the same in principle with that used in other cases, but the form of the work is similar to that of long division of abstract numbers.

```
     £   s.  d.    £   s.  d.
29)721  14  9½  ( 24  17  9 1/58
   58
   141
   116
    25
    20
   514
   29
   224
   203
    21
    12
   261
   261
```

```
days hr. m.  s.
17)47 10  51  17 ( 2  18  59  29 4/17
   34
   13
   24
  322
  17
  152          8
  136         60
   16        497
   60        34
 1011        157
  85         153
  161          4
  153
    8
```

If the divisor be 100, 1000, 10,000, &c., the division at each stage of the process may be most quickly performed by cutting off two, three, or four figures to the right.

```
  £      s.   d.
12,73    9    6¼ ÷ 100
   20
14,69       34¼ = 137/4
   12
 8,34

£12 14 8 137/400
```

```
 tons cwt. qr. lb.  oz.
3)24   1   3   20   9 ÷ 3000
   8   0   2   16   3
  20
 160
   4
 642
  28                  lb.  oz.
                      17   14   14
5152
1284
17,992
    16
 5955
 992
14,875
    16
 5250
 875
14,000
```

Ex. 9.

Find the values of:—

1.	£31 5*s.* $7\frac{1}{2}$*d.* ÷ 6.	2.	£52 18*s.* 2*d.* ÷ 8.
3.	£73 2*s.* $8\frac{1}{4}$*d.* ÷ 9.	4.	£87 12*s.* $5\frac{1}{4}$*d.* ÷ 11.
5.	£93 15*s.* 7*d.* ÷ 10.	6.	£15 19*s.* $2\frac{1}{2}$*d.* ÷ 7.
7.	£23 5*s.* 9*d.* ÷ 8.	8.	£16 13*s.* $11\frac{3}{4}$*d.* ÷ 12.
9.	3 cwt. 1 qr. 2 lbs. 3 oz. ÷ 7.	10.	16 lb. 7 oz. 13 dwt. 13 gr. ÷ 5.
11.	5 qr. 6 lb. 11 oz. ÷ 9.	12.	37 mi. 3 fur. 190 yd. 1 ft. ÷ 12
13.	£147 13*s.* 6*d.* ÷ 48.	14.	£235 16*s.* $1\frac{1}{4}$*d.* ÷ 24.
15.	£142 8*s.* $4\frac{3}{4}$*d.* ÷ 35.	16.	2 tons 5 cwt. 3 qr. 7 lb. ÷ 64.
17.	15 yd. 2 ft. 1 in. ÷ 77.	18.	32 lb. 5 oz. 5 dwt. ÷ 240.
19.	£137 16*s.* 5*d.* ÷ 31.	20.	£249 3*s.* $8\frac{1}{2}$*d.* ÷ 19.
21.	£679 16*s.* $8\frac{1}{2}$*d.* ÷ 47.	22.	19 bus. 5 gal. 1 qt. ÷ 37.
23.	46 yrs. 40 d. 10 h. 53 m. 51 sec. ÷ 297.	24.	673 sq. yd. 7 ft. 136 in. ÷ 59.
25.	£36 18*s.* $6\frac{1}{2}$*d.* ÷ 10.	26.	171 cub. yd. 13 ft. 1260 in. ÷ 100.
27.	£1723 18*s.* $9\frac{3}{4}$*d.* ÷ 100.	28.	£272 16*s.* 3*d.* ÷ 400.
29.	£628 1*s.* $9\frac{1}{2}$*d.* ÷ 1000.	30.	£7795 18*s.* ÷ 5000.

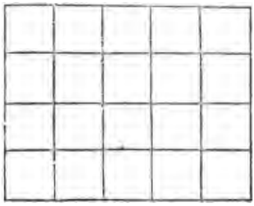
It has now been explained how concrete numbers may be added together, or one subtracted from another, and how they may be either multiplied or divided by abstract numbers.

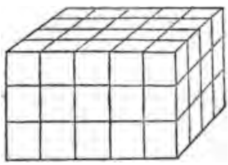
A question now arises with respect to the meaning to be attached to multiplication and division by concrete numbers. Suppose first the case of an abstract multiplied by a concrete number, as for instance 7 multiplied by 8 feet. If our first idea of multiplication be adhered to, this would mean a number of sevens added together, such number being equal to 8 feet, and this meaning is utterly unintelligible. But it has been stated that the full meaning of the words multiplication and division could not at first be given, and we shall find that this case of 7 multiplied by 8 feet leads to a sound and advantageous extension of their signification. In abstract numbers we found it was universally true that the order of the factors was immaterial, so that 7 times 9 and 9 times 7 were each equal to the same number, 63. If this law were extended to the present case, 7 multiplied by 8 feet ought to be the same as 8 feet multiplied by 7, and we know that the latter is 56 feet. It is clear, therefore, that we cannot adhere both to our original definition of multiplication, and to the law that the order of the factors is immaterial. From the definition it follows that 7×8 feet has no meaning, and from the law it follows that 7×8 feet means 56 feet. Being thus obliged to abandon one or the other, it is better to abandon the definition, and extend our meaning of the word multiplication so as to retain the universal application of the law.

Again, in abstract numbers, it was a law that multiplication and division were contrary operations; that is, that a number multiplied and the product then divided by the same number would remain unchanged. Thus $11 \times 12 = 132$ and $132 \div 12 = 11$. If this law be extended, we should have 7 multiplied by 8 feet $=$ 56 feet, and therefore 56 feet, divided by 8 feet, must be 7. Hence an abstract number may be multiplied by a concrete number, giving a concrete number in the product, and a concrete number may be divided by a

concrete number of the same kind, giving an abstract number in the quotient. Before such a division can be performed, both quantities must be expressed in the same denomination; thus £1 8*s.* 2*d.* ÷ 2*s.* 2*d.* = 338*d.* ÷ 26*d.* = 13. There remains next the case of the multiplication of two concrete numbers together; as for instance, 8 shillings by 9 shillings, and to such a multiplication no meaning can in general be assigned.

In the case however where the concrete numbers express length, as for instance, 4 feet × 5 feet, a meaning may be given to the product. It is true that arithmetic, or the science of number, cannot determine this meaning, but Geometry, or the science of space, does determine it. It is a geometrical fact that if there be an oblong figure the sides of which are a certain number of feet or inches in length, and if from the points of division into feet or inches, lines be drawn parallels to the sides, then the oblong will be by these lines divided into a number of squares, the sides of which will be one foot or one inch in length. From arithmetical considerations, it may be seen that the number of the squares must be the product of the numbers of feet and inches in the two sides of the oblong. Hence, combining the geometrical and arithmetical facts together, and calling a square whose side is 1 foot or 1 inch a square foot or a square inch respectively, it follows that 4 feet multiplied by 5 feet may be interpreted to mean 20 square feet.





The above reasoning may be carried one step farther. Let there be a number of cubical blocks of wood, measuring one foot each way. And sup-

pose that these, placed close together and piled upon one another, form a large block 4 feet wide, 5 feet long and 3 feet high. There are then evidently three layers, each of which contains 4 × 5, or 20 blocks. Therefore there must be altogether 3 × 20, or 60 blocks. And as each is one cubic foot, that is, measures one foot in each of three directions, it follows that 3 feet × 4 feet × 5 feet may be interpreted to mean 60 cubic feet.

Since 1 foot = 12 inches, 1 foot × 1 foot, or 1 square foot, must be equal to 12 inches × 12 inches or 144 square inches, and 1 foot × 1 foot × 1 foot, or a cubic foot, must be equal to 12 × 12 × 12, or 1728 cubic inches. Similarly as 1 yard = 3 feet, a square yard = 9 square feet, and a cubic yard = 27 cubic. feet. In this manner the tables of square and cubic measure depend upon the table of measures of length.

The following examples are instances of the application of the above principles :—

Find the area of a room 27 ft. 5 in. long and 14 ft. 3 in. wide.

$$
\begin{array}{rl}
\text{ft.} & \text{in.} \\
27 & 5 \\
\underline{12} & \\
329 & \\
\underline{171} & \\
329 & \\
2303 & \\
\underline{329} & \\
12)\underline{56259} & \\
12)\underline{4688} & 3 \\
9)\underline{390} & 8 \\
43 & 3
\end{array}
\left.\begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array}\right.
\begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \\ 99 \\ \\ \end{array}
\qquad
\begin{array}{rl}
\text{ft.} & \text{in.} \\
14 & 3 \\
\underline{12} & \\
171 &
\end{array}
$$

Area = 43 sq. yd. 3 sq. ft. 99 sq. in.

How many square yards of paper would be required for a room 17 ft. 8 in. long, 11 ft. 2 in. wide, and 10 ft. 4 in. high ?

Here the area of one of the longer walls will be 10 ft. 4 in. × 17 ft. 8 in.; and the area of one of the shorter will be 10 ft. 4 in. × 11 ft. 2 in., and consequently the area of the four walls will be 10 ft. 4 in. × twice the sum of 17 ft. 8 in. and 11 ft. 2 in.

ft.	in.
17	8
11	2
28	10
	2
57	8
12	
692	

ft.	in.
10	4
12	
124	
692	
248	
1116	
744	
85808	

$$\begin{array}{l} 12)\underline{85808} \\ 12)\underline{\ 7150}\quad 8 \\ \ \ 9)\underline{595}\quad 10 \\ \ \ \ \ 66\ \ 1 \end{array}\left.\begin{array}{l} \\ \\ \end{array}\right\} 128$$

	sq. yd.	sq. ft.	sq. in.
Quantity of paper =	66	1	128

How many cubic yards of earth should be taken from a portion of a railway cutting 75 yards in length, the cutting being 33 ft. in width at the bottom, and 57 ft. at the top, and being 24 ft. deep ?

The average width of the cutting will be half the sum of 33 and 57, or 45 ft., and the area of a section of the cutting will be 45 ft. × 24 ft., or 1080 square feet, which is 120 square yards. Hence the number of cubic yards in the portion of the cutting will be 120 sq. yds. × 75 yds. = 9000 cubic yards.

The freight on goods being 36*s.* per ton measurement, a ton being 40 cubic feet, and all fractions of a ton less than $\frac{1}{4}$ ton being charged as $\frac{1}{4}$ ton, what would be the freight on six packages 5 ft. 7 in. long, 3 ft. 4 in. wide, 2 ft. 8 in. deep?

ft. in.	in.
5 7 =	67
	32
	134
	201
	2144
	40
	85760
	6
	514560

3 ft. 4 in. = 40 in. 2 ft. 8 in. = 32 in.

$$\left.\begin{array}{l} 12)\underline{514560} \\ \ \ 12)\underline{42880}-0 \\ \ \ \ \ 12)\underline{3573}-4 \\ \ \ \ \ \ \ 40)\underline{297}-9 \end{array}\right\} 1344$$
$$\qquad\quad 7\ -17$$

The space occupied is therefore 7 tons measurement, and 17 ft. 1344 in., which will be charged as $7\frac{1}{2}$ tons, and $7\frac{1}{2} \times 36s. = 270s.$, or £13 10*s.*

As length × length gives superficial area, and length × length × length gives cubical content, it must follow, from the relation of multiplication to division, that cubical content ÷ area gives length, cubical content ÷ length gives area, and that area ÷ length gives length.

A reservoir has an average area of a quarter of an acre, and in making it, there were taken out 1815 cubic yards of earth. What is its average depth?

Here a quarter of an acre = 1210 sq. yd. and 1815 cub. yd. ÷ 1210 sq. yd. = 1 yd. 1 ft. 6 in. = depth.

A room contains 2730 cubic feet of air, and is 13 feet high. How many square yards of carpet will be required for the floor?

2730 cub. ft. ÷ 13 ft. = 210 sq. ft. = 23 sq. yd. 3 ft.

If a street a mile long cover three acres of ground, how many feet is it wide?

Here 3 acres = 14520 sq. yd. and 1 mi. = 1760 yd.

And 14520 sq. yd. ÷ 1760 yd. = 8 yd. 0 ft. 9 in. = 24 ft. 9 in.

$$
\begin{array}{r}
110)\underline{1452.0} \\
4)\underline{132} \\
4)\underline{33} \\
8\tfrac{1}{4}\text{ yd.} = 8\text{ yd. } 0\text{ ft. } 9\text{ in.}
\end{array}
$$

Many questions in arithmetic, involving the multiplication of length by length, or the division of area or cubical space by length, are most easily worked out by fractions, and therefore the above and following instances are given more for the purpose of illustrating principles than of showing the best methods of working the questions proposed.

Ex. 10.

Find the values of:—

1. £1 13*s.* 4*d.* ÷ 6*s.* 8*d.*
2. £5 0*s.* 9*d.* ÷ 7*s.* 9*d.*
3. £211 2*s.* $0\frac{3}{4}$*d.* ÷ £4 18*s.* $2\frac{1}{4}$*d.*
4. £445 6*s.* $7\frac{1}{2}$*d.* ÷ £5 9*s.* $11\frac{1}{2}$*d.*
5. 13 cwt. 1 qr. 25 lb. 2 oz. 11 dr. ÷ 13 lb. 5 oz. 11 dr.
6. 70 d. 4 h. 44 m. 5 sec. ÷ 11 h. 18 m. 25 sec.
7. Divide the sum of 8 cwt. 1 qr. 3 lb. and 4 cwt. 19 lb. by the difference between 4 cwt. 2 qr. 15 lb. and 3 cwt. 3 qr. 17 lb.
8. Multiply the product of 18 and 27 oz. avoirdupois by the quotient of 8 acres 1 rood ÷ 2 roods 8 poles.

Find the values of:—

9. 8 ft. 5 in. × 3 ft. 7 in.
10. 11 ft. 6 in. × 14 ft. 5 in.
11. 13 ft. 2 in. × 11 ft. 1 in.
12. 17 ft. 3 in. × 25 ft. 7 in.
13. 26 ft. 5 in. × 13 ft. 7 in.
14. 42 ft. 3 in. × 11 ft. 7 in.
15. 26 sq. yd. 5 ft. 109 in. ÷ 2 ft. 1 in.
16. 4 sq. yd. 3 ft. 72 in. ÷ 2 yd. 7 in.

17. 68 sq. yd. 4 ft. 99 in. ÷ 5 yd. 2 ft. 3 in.

18. 173 sq. yd. 4 ft. 56 in. ÷ 14 yd. 7 in.

19. 3 ft. 7 in × 8 ft. 4 in. × 2 ft. 9 in.

20. 15 ft. 7 in. × 9 ft. 5 in. × 8 ft. 7 in.

21. The floor of a room is 18 ft. 6 in. long, and 13 ft. 6 in. wide. What is its area, and what length of carpet 2 ft. 3 in. wide would be required to cover it?

22. A room is 23 ft. 1 in. long, 17 ft. 5 in. broad, and 12 ft. 2 in. high. What will be the total cost of paper for the walls at $2\frac{1}{4}d.$, oil-cloth for the floor at 1*s.* 9*d.*, and whitewash for the ceiling at 1*d.* per square yard, any fraction of a square yard being in each case charged as a square yard?

23. If railway sleepers are 5 ft. long, $4\frac{1}{2}$ in. broad, and $3\frac{1}{2}$ in. thick, how many cubic feet of wood are there in every hundred of them?

24. What is the depth of a trench 2 ft. 4 in. broad, if in making 18 chains of it there are taken out 154 cubic yards of earth?

MISCELLANEOUS EXAMPLES ON CHAPTERS I. AND II.

1. Multiply 93747 by 829 and divide 599497 by 487.

2. Add together £8 2*s.* $3\frac{1}{2}d.$, £10 12*s.* $7\frac{3}{4}d.$, £6 0*s.* $1\frac{1}{2}d.$, £15 7*s.* $2\frac{1}{4}d.$; and 3 cwt. 2 qr. 6 lb. 9 oz., 4 cwt. 3 qr. 15 lbs. 7 oz., 6 cwt. 1 qr. 23 lb. 14 oz., 8 cwt. 2 qr. 16 lb. 9 oz.

3. Divide £1311 18*s.* $9\frac{1}{2}d.$ by 17.

4. Divide 17 acres 2 roods of land equally among 22 persons.

5. How many minutes of time are there between 20 minutes past 4 P.M. August 27th, and 13 minutes before 6 A.M. November 7th of the same year?

6. Multiply 827963 by 347, and divide 769437 by 229.

7. Reduce £12 10*s.* 8*d.* to fourpenny pieces, and 14352 guineas to half-crowns.

8. A man drinks one quart of beer, and his wife one pint every day, and each one of five children drinks on an average half a pint twice a week. What is the weekly consumption of beer, how much does it average daily, and how long would it take them to finish a cask containing 9 gallons 3 quarts?

9. Find the value of 127 yards of calico at $4\frac{3}{4}d.$ per yard.

10. The divisor being 872, the quotient 1134, and the remainder 563, find the dividend.

11. A piece of ground containing 19 acres 2 roods 27 perches

$24\frac{1}{4}$ yards is to be divided into allotments, each containing 1 rood 11 perches $18\frac{1}{4}$ yards. How many will there be?

12. If a bale of wool containing 4 cwt. 2 qr. 20 lb. sell for £68 15*s*. 6*d*., how much is that per lb.?

13. What is the area of a room 16 ft. 6 in. long, and 10 ft. 3 in. broad, and what length of carpet 27 in. wide would be required for it?

14. Reduce 3 tons 7 cwt. 11 lb. to ounces, and 7843291 seconds to weeks.

15. Multiply £13 15*s*. $2\frac{3}{4}$*d*. by 35, and divide £816 15*s*. $7\frac{1}{2}$*d*. by 59.

16. A number is multiplied by 4, 8 is added to the product, and the sum being divided by 12 gives as a quotient 3. Find the number.

17. A watch is set right at noon on the 9th of August. If it is $15\frac{1}{2}$ minutes too fast at 6 P.M. on the 1st of September, how many seconds has it gained per day?

18. Two men at 5*s*. and a boy at 2*s*. 6*d*. a day were employed in putting up the framework of a wooden house, in which were used 1200 ft. of timber at 23*s*. per hundred feet, and 120 lb. of nails and bolts at an average price of $2\frac{1}{2}$*d*. per lb. The total cost being £21 6*s*., find how long they took to do the work.

19. If a man walk at the rate of 110 yards a minute, and for 4 hours a day, how many miles will he go in 27 days?

20. Find the difference between the value of a million farthings and 209 five pound notes.

21. Reduce 8974 grains troy to lbs., and 67893 pints to loads.

22. What will it cost to paper a room 18 ft. 6 in. long, 13 ft. 9 in. broad, and 10 ft. 6 in. high, with paper 27 in. wide, at $4\frac{1}{2}$*d*. per yard?

23. Reduce 763254 drams to tons, and 213762975 seconds to years.

24. Multiply £25 2*s*. 1*d*. by 36, and divide the product by 75.

25. A lady buys velvet at 9*s*. 4*d*., and half as much silk at 4*s*. $10\frac{1}{2}$*d*., and altogether spends £8 4*s*. $9\frac{1}{2}$*d*. How much of each does she buy?

26. Find the number of acres in a square space each side of which is 700 feet.

27. A wine merchant mixes 100 gallons of sherry at 14*s*. with 30 gallons at 20*s*. At what price must he sell the mixture so as to gain 6*s*. on every £1 of his outlay?

28. What is the value of 7 boxes of tea, each containing 14 lb. 12 oz. at 3*s*. 2*d*. per lb.?

29. To each of several poor men is given at Christmas a purse containing a sovereign, a crown, a shilling, a sixpence, a penny, a halfpenny, and a farthing. The whole amount paid being £30 12*s*. $10\frac{1}{4}$*d*., what was the number of poor men?

30. The water in a mill-lead has an uniform width of 15 and depth

of $3\frac{1}{2}$ inches. If it flows at the rate of $1\frac{1}{2}$ miles an hour, how many gallons will be discharged in a day, 10 gallons being taken to contain 2772 cubic inches?

31. Divide £36 4*s.* 3*d.* among 4 persons, so that the first may have £2 5*s.* 3*d.* more than the second, the second £3 1*s.* 4*d.* more than the third, and the third £2 12*s.* 6*d.* more than the fourth.

32. A book contains 232 pages of 36 lines, consisting on the average of 14 words each. How long would it take a man to make a copy of it, supposing he could write in an hour 112 lines of 12 words each, and worked 6 hours a day?

33. Multiply £18 13*s.* $4\frac{1}{2}$*d.* by 13, and divide the product by 29.

34. If a gentleman can save £27 out of every £100 he receives, and in 3 years he has thus saved £425 5*s.*, how much has he spent per day during this period?

35. Reduce 2 acres 1 rood 17 perches 3 yards to square feet.

36. Divide £25 16*s.* $0\frac{1}{4}$*d.* among 10 men and 11 women, giving each man twice as much as each woman.

37. Find the number of ells in 645 poles, and the number of ounces avoirdupois in 875 dwts. troy.

38. What will be the expense of papering a room 19 ft. 4 in. long, 16 ft. 8 in. broad, and 12 ft. high, with paper a yard wide, and costing 16*d.* per piece of 12 yards?

39. Sound travels at the rate of 1140 feet per second. What interval of time will there be between the flash and report of a gun fired $2\frac{1}{2}$ miles off?

40. Multiply 17 cwt. 1 qr. 12 lb. 9 oz. 13 dr. by 28.

41. The wards of a hospital are 12 feet high, allowing 600 cubic feet of air to each patient, and 500 yards of matting 3 feet 6 inches wide are laid down between the beds, altogether covering a quarter of the floor. Find how many patients the hospital can accommodate.

42. A gentleman living in London, where there is no Sunday post, spends on the average 1*s.* 3*d.* per day on postage. What does he spend in this way in 28 years, regard being had to Leap years?

43. A spirit merchant mixes brandy at 24*s.* per gallon, with 12 gallons of inferior spirit at 8*s.* If he gains £15 18*s.* by selling the mixture at 26*s.* 4*d.* per gallon, find what was the original quantity of brandy.

44. The first of three partners in business receives a share half as much again as that of the second, and twice that of the third. How should £1300 be divided between them?

45. If goods are bought at £3 3*s.* per cwt., and the cost of carriage is £1 5*s.* per ton, and they are sold at 8*d.* per lb., what is the profit on each quarter?

46. A man buys 126 dozen of apples, partly at $3d.$ and partly at $4d.$ a dozen. He sells them all for £2 12s. 6d. and finds that he has gained at the rate of 10d. per 6 dozen. How many are there of each kind?

47. On the Metropolitan Railway, trains run each way every quarter of an hour, from 6 A.M. to 12 P.M. Supposing the average number of passengers to be 150, and the average fare paid $3\frac{1}{4}d.$, what would be the weekly receipts?

48. Supposing 10 gallons to contain 2772 cubic inches, how many gallons of beer would there be on the cooling floor of a brewery 22 yards long and 7 yards broad, if the depth of beer were $2\frac{1}{2}$ inches?

49. After paying income tax at the rate of 7d. in the £ a gentleman has £533 19s. 2d. remaining, what amount of tax did he pay?

50. What is the area of a gravel walk 4 ft. 8 in. wide, which encloses a grassplot 80 yards long and 35 yards broad?

CHAPTER III.

ABSTRACT NUMBERS CONTINUED, AND FRACTIONS.

In Chapter I. Abstract Numbers were sufficiently treated of to render their first applications to practical purposes intelligible, and these first applications were contained in Chapter II. It is now necessary to return to the subject of Abstract Numbers, and to consider some of their properties which have not hitherto been explained, but which must be understood before the more advanced parts of Arithmetic, and especially the nature and use of fractions, can be proceeded with.

Taking as an instance the number 1425, it has been already shown how it should be expressed either in words or figures, what is signified by each of the figures composing the number, and how it may be either added to or subtracted from another number, or used to multiply or divide any numerical quantity, abstract or concrete. But there is something more that may be known about it. Let it be divided by 5, then $1425 \div 5 = 285$, or $1425 = 5 \times 285$. Let 285 be divided by 5, then $285 = 5 \times 57$. Let 57 be divided by 3, then $57 = 3 \times 19$. Combining these results, we should have $1425 = 5 \times 5 \times 3 \times 19$, and thus four factors have been found whose product $= 1425$. It is not every number that can be so separated into factors; some are not divisible by any numbers except themselves and unity, and such are called *prime numbers*. Thus 23, 41, 59, are prime numbers. Those numbers, on the other hand, that can be separated into factors, are called *composite numbers*, and any composite number may be called a *multiple* of any of its factors.

Thus 28 is a composite number, and a multiple of 4, 7, and 14. To separate a number into as many factors as possible, care must be taken that none of the factors are themselves composite numbers, and the number is then said to be separated into its *prime factors*. This process is very useful, because by its aid calculation is frequently shortened to a very considerable extent, at the same time, it is not absolutely necessary, inasmuch as all operations might be performed without reference to it, the work being, indeed, often excessively long and cumbersome, in place of short and easy, but nevertheless being quite certain and accurate. It may be observed also, that there is no general rule known, by which any number may be separated into its factors. Practically, however, though a perfect knowledge of this subject is perhaps impossible, enough is known, and can be made the subject of rules and processes, which are true as far as they go, to afford us nearly the same advantages as would result from our knowing it thoroughly.

It is always easy to determine whether any number is divisible without remainder by any of the numbers from 2 to 12 inclusive, with the exception of 7. The following are the rules for this purpose:—

1. A number is divisible by 2, if the last figure is divisible by 2. Thus 87624 is divisible by 2, because 4 is. Any number divisible by 2 is termed an *even number*, and a number not so divisible is called an *odd number*.

II. A number is divisible by 4 or 8, if the number formed by the last two or last three figures respectively is divisible by 4 or 8. Thus 87624 is divisible by 4, because 24 is divisible by 4. And 87624 is divisible by 8, because 624 is divisible by 8.

III. The figures composing a number are called its *digits*. A number is divisible by 3 or 9, if the sum of its digits is divisible by 3 or 9. Thus 87624 is divisible by either 3 or 9, the sum of its digits being 27. And 73842 is divisible by 3, but not by 9, the sum of the digits being 24.

IV. A number is divisible by 5, if the last digit is 5 or 0.

V. A number is divisible by 11, if the sum of the first, third, fifth, &c. digits is either equal to the sum of the second, fourth, sixth, &c. digits, or if the difference between these two sums is 11, or a multiple of 11. Thus 37598 is divisible by 11, the sum of 3, 5, and 8 being 16, and that of 7 and 9 being also 16. Again, 49281705 is divisible by 11, the sum of the first, third, &c. or *odd digits* being 7, and the sum of the *even digits* being 29, the difference of these sums being consequently 22.

By means of these rules, all the factors of a number not greater than 12 may be discovered, with the exception of 7, the rule respecting that number being too complicated to be of much service. (Note F.) A number will obviously be divisible by 6, if it is divisible by both 3 and 2; it will be divisible by 12, if by both 3 and 4, and by 10 if the last digit is 0.

As an example of separating a number into its factors, let 1441440 be the number. One method of proceeding will be to see first whether 2, 4, or 8 will divide it. It appears that 8 will, since 440 is divisible by 8. The quotient is 180180, which is divisible by 4, but not by 8, giving as quotient 45045. Adding the digits together, we find that 9 divides this, giving 5005. As the sum of the digits of 5005 is 10, neither 3 nor 9 will divide it, and a new number must be tried. 5 obviously is a factor giving as quotient 1001. Next, as the sums of the odd and even digits are equal, 11 divides this, giving as quotient 91. Lastly, 7 must be tried, and $91 \div 7 = 13$. Hence the number has been separated into its factors, and $1441440 = 8 \times 4 \times 9 \times 5 \times 11 \times 7 \times 13$. Of these factors 8, 4, and 9 are themselves *composite numbers*, that is, the product of certain factors, $8 = 2 \times 2 \times 2$; $4 = 2 \times 2$; $9 = 3 \times 3$. Hence we have that

$$1441440 = 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 11 \times 7 \times 13.$$

A shorter method of writing this, is to put above, and a little to the right of each factor that occurs more than once, a figure indicating the number of times it occurs, thus, $1441440 = 2^5 \times 3^2 \times 5 \times 11 \times 7 \times 13$.

The expression 2^5 is read '*Two to the fifth power,*' and the small figure 5 is called the *index*. Instead of the second and third powers, other words are often used. Thus 3^2 is called 3 *squared*, or *the square of* 3, and 3^3 would be called 3 *cubed*, or *the cube of* 3.

Supposing that a number has no factors less than 12, it is generally more or less difficult, and often impossible, to separate it into its prime factors. As a first instance, take the number 167. Now, $12 \times 12 = 144$, and $13 \times 13 = 169$. It is obvious, therefore, that if 167 has any factors at all, one of them must be less than 13, because if they were both equal to or greater than 13, their product would be greater than 167. Now we can immediately see that 167 is not divisible by any number as far as 12. Consequently, 167 is a prime number. We could in this way determine the prime numbers as far as 167. The following is a list of them as far as 101 :—2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101. If, therefore, in the case of any given number it appeared that there was no factor less than 12, it would be necessary to try 13, 17, 19, 23, &c., in succession, until we had arrived at a number such that, when multiplied by itself, the product was equal to or greater than the number in question. The following examples are illustrations of this point :— 521 is a prime number, but to determine this we must try as far as 19, not trying 23, because $23 \times 23 = 529$. In the case of 727, also a prime number, we should try 23, and not 29. In the case of 437 we should find 19 was a factor, and that $437 = 19 \times 23$.

Similarly, $713 = 23 \times 31$; $7429 = 17 \times 19 \times 23$; $846560 = 2^5 \times 5 \times 11 \times 13 \times 37$. (Note G.)

Ex. 11.

1. Give a list of the prime numbers between 101 and 167.

2. Give a list of the successive powers of 2 as far as the twelfth, of 3 as far as the eighth, of 5 as far as the sixth, and of 7 and 11 as far as the fourth.

3. Give a list of the squares of all the prime numbers from 13 to 31.

Separate into their factors —

4. 84.	5. 96.	6. 360.	7. 336.
8. 11880.	9. 944.	10. 2268.	11. 5640.
12. 15675.	13. 5291.	14. 12397.	15. 30030.
16. 41503.	17. 11205.	18. 51480.	19. 69632.

Determine whether the following numbers are prime, if not, give their factors, and if prime, state what is the highest divisor that it is necessary to try.

20. 353. 21. 401. 22. 437. 23. 563. 24. 617. 25. 667.

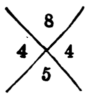
Separate into their factors —

26. 2193. 27. 6726. 28. 4991.

Before proceeding to use this process of separation into factors, a digression will be made for the purpose of explaining one important application of the property of the number 9 that has just been mentioned. A number is divisible by 9 when the sum of its digits is so divisible. From this it may be shown that any number when divided by 9, and the sum of its digits when divided by 9, both leave the same remainder. Thus the sum of the digits of 284 is 14, and $284 \div 9 = 31$ and 5 over, while $14 \div 9 = 1$ and 5 over. Again, $437286 \div 9 = 48587$ and 3 over, while the sum of the digits $30 \div 9 = 3$ and 3 over.

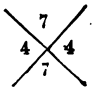
Suppose now that two numbers are multiplied together, as, for instance, $431 \times 257 = 110767$. Now, $431 = 47$ nines and 8; $257 = 28$ nines and 5. And $431 \times 257 = 47$ nines and 8, all multiplied by 257, which must be equal to some number of nines and 8 times 257. But 8 times $257 = 8$ times 28 nines and 5; that is, some number of nines and 40. And $40 = 4$ nines and 4. Hence whatever the product may be, we see that it ought, when divided by 9, to leave a remainder 4. And this furnishes a test of the

correctness of the work. If the product 110767, or, what has been shown to be the same thing, the sum of its digits, leaves the remainder 4 when divided by 9, there is a strong probability of the result being correct. Now, the sum of the digits is 22, and 22 = 2 nines and 4. This proof of the result of a multiplication is called the proof by *casting out the nines*, and, for the sake of shortness, the remainder, after dividing the sum of the digits by 9, may be called the *digits remainder*. The usual way of applying this method to prove a multiplication is as follows :—The digits remainder of the multiplier is placed in the upper space of a cross, that of the multiplicand in the lower space, that of the product of these two digits remainders in one of the side spaces; and should this last agree with the digits remainder of the product of the original multiplication, the work is probably correct. In the example just given the proof would stand thus :—

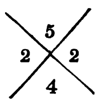


As additional examples the two following multiplications, with their proofs, are given :—

```
   4876
    547
  34132
 19504
24380
2667172
```



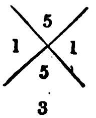

```
  36824
    157
 257768
184120
36824
5781368
```



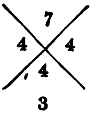
The same kind of proof is also applicable to an example in division. The long and regular proof would be to multiply the quotient by the divisor, add the remainder, and the result ought to be the dividend. In the proof by 'casting out the nines' we substitute for divisor, quotient, remainder, and dividend, their respective digits remainders. The first may be put in the upper space of the cross, the second in the lower, the third a little distance below that. Then, multiplying the first and second together, and adding the third, put the digits remainder of the result in the space on one side, and this should be the same as that of the

dividend, to be placed in the remaining space. The following are two examples, with proofs :—

```
734)208936(284
    1468
    ----
     6213
     5872
     ----
      3416
      2936
      ----
       480
```




```
376)184297(490
    1504
    ----
     3389
     3384
     ----
       57
```



The proof by casting out the nines is not absolutely certain, but it gives a strong probability of correctness, and from its being so short it is practically of great use. The student should not, however, be satisfied till he can use it, either in testing a multiplication or a division, without writing anything down at all. This may easily be managed by a little practice, and after a time it will be found that the proof may be applied almost at a glance. The advantage of being able thus to check the accuracy of a long calculation at every step of the process is obvious.

Returning from this digression to the main subject, there are two processes which must be understood before Fractions can be fully treated upon. The first of these has for its object to find the greatest number that will divide two or more numbers without a remainder. This number is called their *Greatest Common Measure*, and is often briefly expressed by the initial letters as their G. C. M. To explain this process, it must be premised that if a divisor and a dividend both contain a certain factor, the remainder must also contain that factor. Thus 320 and 112 both contain the factor 16. $320 \div 112 = 2$ and 96 over, and $96 =$ six times 16. Again, 350 and 925 both contain the factor 25 ; $925 \div 350 = 2$ and 225 over. And $225 = 9$ times 25.

Let it now be required to find the G. C. M. of 10080 and 3696. This being a factor of both must be a factor of the remainder left after dividing the larger by the smaller. $10080 \div 3696 = 2$ and 2688 remainder. It must therefore be a factor of 2688, and as it must also be a factor of 3696,

and obviously the greatest possible factor of these, it must be the G. C. M. of 3696 and 2688. Now 3696 ÷ 2688=1 and 1008 remainder, and therefore by reasoning similar to that already given, we see that the question is reduced to that of finding the G. C. M. of 2688 and 1008. 2688 ÷ 1008=2 and 672 remainder. We have therefore to find the G. C. M. of 672 and 1008. 1008÷672=1 and 336 over. The original question is therefore now replaced by that of finding the G. C. M. of 672 and 336. But this is at once answered, as 336 divides 672 exactly, and 336 must therefore be their G. C. M. Consequently 336 is also the G. C. M. of 10080 and 3696. The general rule for finding the G. C. M. of two numbers may be stated thus: *Divide the larger by the smaller and then the divisor by the remainder, afterwards continue dividing each successive divisor by the corresponding remainder until there is no remainder, the last divisor is the G. C. M.*

The instance given above may be written down in the following manner:—

```
3696)10080(2
      7392
      ----
      2688)3696(1
           2688
           ----
           1008)2688(2
                2016
                ----
                 672)1008(1
                      672
                      ---
                      336)672(2
                          672
                          ---
```

This mode of writing down the work is, however, not so short and compact as it might be, if we left out the successive quotients which are not wanted, and did not put down each remainder twice over. This example and three others, namely 2576 and 8848; 4900 and 19712; and 2079 and 1024 are here given in the shorter form. (Note H.)

3696	10080	2576	8848	4900	19712	1024	2079
2688	7392	2240	7728	448	19600	93	2048
1008	2688	336	1120	420	112	94	31
672	2016	336	1008	336	84	93	31
336	672		112	84	28	1	
	672			84			
G. C. M. = 336.		G. C. M. = 112.		G. C. M. = 28.		G. C. M. = 1.	

In the last of these examples, where the G. C. M. is 1, the numbers are said to be *prime to one another*, that is, they have no common divisor. It must be noticed that this is a very different thing from their being prime numbers. In fact neither of the numbers 1024 and 2079 is a prime number, since $1024 = 2^{10}$ and $2079 = 3^3 \times 7 \times 11$.

If it be required to find the G. C. M. of several numbers a general rule is to find the G. C. M. of two, then of that G. C. M. and a third, again of that second G. C. M. and a fourth, and so on. Thus, to find the G. C. M. of 1848, 900, 630, and 4096. The G. C. M. of 1848 and 900 is 12; that of 12 and 630 is 6; that of 6 and 4096 is 2, and therefore 2 is the G. C. M. of the four quantities originally given. It is, however, seldom, if ever, required practically to find the G. C. M. of more than two numbers.

If the numbers, whether two or more, can be easily separated into factors, the G. C. M. will be the product of the prime factors, or powers of prime factors, that occur in every one of the numbers. Thus, taking the instance just worked out: $1848 = 2^3 \times 3 \times 7 \times 11$; $900 = 2^2 \times 3^2 \times 5^2$; $630 = 2 \times 3^2 \times 5 \times 7$; $4096 = 2^{12}$. Here the only factor that occurs in all is 2, and in one case, namely 630, it appears only in the first power. Hence the G. C. M. is 2 to the first power, or 2.

Again, to find the G. C. M. of 336, 812, 1400—

$336 = 2^4 \times 3 \times 7$; $812 = 2^2 \times 7 \times 29$; $1400 = 2^3 \times 5^2 \times 7$; therefore G. C. M. $= 2^2 \times 7 = 28$.

The four examples of two numbers given above would by the method of separation into factors be worked out as follows—

$3696 = 2^4 \times 3 \times 7 \times 11$; $10080 = 2^5 \times 3^2 \times 5 \times 7$; therefore G. C. M. $= 2^4 \times 3 \times 7 = 336$; $2576 = 2^4 \times 7 \times 23$; $8848 = 2^4 \times 7 \times 79$; therefore G. C. M.

$=2^4 \times 7 = 112$; $4900 = 2^2 \times 5^2 \times 7^2$; $19712 = 2^8 \times 7 \times 11$; G. C. M. $= 2^2 \times 7 = 28$; $1024 = 2^{10}$; $2079 = 3^3 \times 7 \times 11$; G. C. M. $= 1$.

Ex. 12.

Find the G. C. M. of

1.	16614 and 71676.	2.	4650 and 6825.
3.	10608 and 32123.	4.	4880 and 7896.
5.	51000 and 123125.	6.	37688 and 113106.
7.	9316 and 21509.	8.	8029 and 17353.
9.	161875 and 362637.	10.	22880 and 40480.
11.	216000 and 727488.	12.	45056 and 69632.

Show that the following pairs of numbers are prime to each other.

13.	705 and 1508.	14.	2717 and 22030.
15.	27040 and 43659.	16.	6961 and 9976.

Find the G. C. M. of

17. 308, 392, 420, and 476.
18. 1035, 225, 540, and 360.
19. 475, 600, 325, and 275.
20. 108, 176, 162, and 180.

The process for finding the G. C. M. was stated to be the first of two that were required for the full consideration of fractions. The second, which we now come to, has for its object to find the least number which certain given numbers will divide without a remainder. Such a number is called their *Least Common Multiple*, and is often briefly mentioned as their L. C. M. The best method of finding this is by separating the numbers into their factors, when *the L. C. M. will be the product of all the factors, each raised to the greatest power to which it occurs in any of the given numbers.* It is true that another method is often given for finding the L. C. M., and in some cases it may be at first a little shorter than that by separation into factors. But the great advantage of the latter over the former is, that by practice it leads to so rapid a mode of working, that frequently the L. C. M. of several numbers may be determined at once by inspection. It also helps to encourage that intimate acquaintance with numbers, and that knowledge of their individual properties and peculiarities, which are characteristic of a good arithmetician.

Find the L. C. M. of 25, 28, 40, 105, 72, and 56.

Here $25=5^2$; $28=2^2\times 7$; $40=2^3\times 5$; $105=3\times 5\times 7$; $72=2^3\times 3^2$; $56=2^3\times 7$.

Hence L. C. M. $=2^3\times 3^2\times 5^2\times 7=12600$.

In this example the whole of the work, except the actual multiplication, has been written down. But practically the way in which the question would be considered would be this: 8 is the highest power of 2 that divides any of the given numbers; 9 is the highest power of 3; 25 is the highest power of 5; and 7 of 7; and there are no other factors. Therefore the L. C. M. is $8\times 9\times 25\times 7=12600$. A little practice would enable the pupil to work out this, or a similar example, without putting down a single figure except the final result. In multiplying the numbers together finally, it should be noticed that multiplication by 25 is the same as division by 4 and multiplication by 100, and that multiplication by 125 is the same as division by 8 and multiplication by 1000. Thus, in the case just given, $8\times 9\times 25\times 7$, we know that $25\times 8=200$. Now $2\times 9=18$ and $18\times 7=126$. Hence the result is 12600. All this work might be done in the mind, without putting down a figure.

Find the L. C. M. of 15, 60, 28, 48, 56, 84.

Here the highest power of 2 is 16 in the number 48, highest of 3 is 3 in several numbers, highest of 5 is 5 in two numbers, highest of 7 is 7 in three numbers. There are no other factors, and hence L. C. M. $=16\times 3\times 5\times 7=1680$.

If any of the given numbers are factors of others, they may be at once struck out and dismissed from consideration; as the multiple of the larger number must necessarily be a multiple of the factor also. Thus in the last example, 28 which is a factor of 56, and 15 which is a factor of 60, might have been at once struck out.

Ex. 13.

Find the L. C. M. of

1. 8, 9, 6, 24.
2. 2, 15, 12, 20.
3. 7, 8, 9, 10.
4. 5, 7, 9, 11.

5. 5, 8, 12, 18, 20.
6. 4, 6, 27, 18, 24, 30.
7. 24, 9, 11, 18, 30, 45.
8. 22, 81, 3, 20, 11.
9. 6, 9, 12, 28, 21, 36, 24.
10. 75, 63, 12, 14, 56.
11. 162, 108, 72, 48.
12. 26, 12, 35, 91, 14.
13. 187, 221, 143, 34, 26.
14. The first ten numbers.
15. The numbers from 11 to 20 inclusive.
16. The odd numbers as far as 19.
17. The even numbers as far as 20.
18. Explain why the answer to 17 is double of the answer to 14.
19. Explain why the L. C. M. of the first 20 numbers is the same as that of the numbers from 11 to 20 inclusive, and 16 times as great as that of the odd numbers as far as 19.
20. Three men run round a field in 5 minutes 12 seconds; 4 minutes; and 3 minutes 15 seconds respectively. If they continued at the same rate, in what time would they again be together at the starting point?

The two processes for finding the G. C. M. and the L. C. M. having been explained, the consideration of fractions may now be entered upon, and it will be well, in the first place, to recapitulate what has been already said about them in a former chapter. A fraction was defined to be the result of dividing a less number by a greater, which result can be expressed in no simpler form than by writing down the two numbers with the sign of division thus, $\frac{3}{7}$. The upper of these numbers is called the *numerator*, and the lower the *denominator*. An expression, fractional in form, but where division is either completely or partially possible, owing to the upper number being greater than the lower, is called an *improper fraction*. Improper fractions, by performing this division, may be always reduced to *whole or mixed numbers*. Thus $\frac{100}{25}=4$, $\frac{23}{7}=3\frac{2}{7}$, and, conversely, all mixed numbers may be reduced to improper fractions by multiplying the whole number by the denominator of the fraction and adding in the numerator. Thus $5\frac{5}{7}=\frac{40}{7}$. Whole numbers may be expressed as improper fractions with any denominator whatever, by taking as numerator the product of the whole number and that denominator. Thus $8=\frac{8}{1}=\frac{24}{3}=\frac{800}{100}=\frac{232}{29}$.

It was shown that a fraction might be multiplied, either by multiplying its numerator or dividing its denominator, and that it might be divided either by dividing its numerator or multiplying its denominator, and that consequently the multiplication or the division of both numerator and denominator by the same number would leave it unaltered.

$$\tfrac{45}{56}\times 7=\tfrac{315}{56} \text{ or } \tfrac{45}{8};\ \tfrac{45}{56}\div 9=\tfrac{5}{56} \text{ or } \tfrac{45}{504};\ \tfrac{45}{56}=\tfrac{450}{560}=\tfrac{495}{616}=\tfrac{225}{280}.$$

And since $\frac{5}{8}$ multiplied by $7=\frac{35}{8}$, we conclude that 7 multiplied by $\frac{5}{8}$ must likewise $=\frac{35}{8}$, and hence we deduce the rule that *multiplication by a fraction means multiplication by its numerator and division by its denominator.* Moreover, since multiplication and division are contrary operations, so that the one counteracts the effect of the other it follows that *division by a fraction means division by its numerator and multiplication by its denominator*, or in other words, multiplication by the fraction inverted.

The above is a recapitulation of what is more fully explained in pages 19 to 23, but there is another view of the nature of fractions to which it is necessary to refer. We know that $\frac{6}{7}=6\times\frac{1}{7}$, similarly $\frac{13}{45}=13\times\frac{1}{45}$, $\frac{17}{84}=17\times\frac{1}{84}$, and so on. We might therefore consider $\frac{6}{7}$ as being 6 of such quantities as would arise from dividing unity by 7, $\frac{13}{45}$ as 13 quantities each equal to $1\div 45$, $\frac{17}{84}$ as 17 times as much as $1\div 84$. Now, in our language, the ordinal numbers are used to represent these imaginary parts of unity. Thus $\frac{1}{7}$, instead of being called 1 divided by 7, may be spoken of more briefly as *one seventh*; $1\div 45$ as *one forty-fifth*; $1\div 84$ as *one eighty-fourth*; and the above fractions would be read *six sevenths*; *thirteen forty-fifths*; *seventeen eighty-fourths.*

This view of fractions explains moreover how the upper number came to be called the numerator and the lower the denominator. The upper is the 'numberer' of the parts of which the lower points out the 'denomination,' that is to say, by what number unity has been divided to produce them.

Both modes of defining the meaning of a fraction are perfectly accurate, but it will probably cause much confusion in the pupil's mind, if he is perplexed with thinking about both at the same time. Throughout this book, therefore, the definition that a fraction expresses the result of dividing the numerator by the denominator will be adhered to; and when the pupil has once seen what is meant by such expressions as twenty thirds, forty sixths, and the like, he may dismiss from his mind all consideration of these imaginary parts of unity.

The subject of Fractions will now be continued. It has been seen (page 21) that the same result may be expressed by an infinite number of fractions. Thus, $\frac{3}{4} = \frac{6}{8} = \frac{60}{80} = \frac{75}{100} = \frac{39}{52}$, &c. Practically, out of all these possible forms the most convenient should be chosen, and this will be when the numerator and denominator are the lowest possible. Such a fraction is arrived at by dividing numerator and denominator by any common factor they may contain. When this has been done to as great an extent as possible, the fraction is said to be in its *lowest terms;* and inasmuch as numerator and denominator have then no common factor, they are *prime to one another*. (See page 59.) When the common factors are easily to be discovered by the tests of divisibility given on page 52, these should be used as divisors in succession. Thus

$$\frac{8192}{17600} = \frac{1024}{2200} = \frac{128}{275} \qquad \frac{4356}{7128} = \frac{1089}{1782} = \frac{121}{198} = \frac{11}{18}$$

When the common factors are not easily to be discovered, we must fall back upon a longer but never-failing method—that is, to divide both numerator and denominator by their G. C. M. Thus, in the case of the fraction $\frac{4169}{5306}$, the G. C. M. is found to be 379, and by dividing both numerator and denominator by this number, the fraction is reduced to $\frac{11}{14}$.

```
4169  5306     379)4169(11     379)5306(14
3411  4169         379             379          4169   11
----  ----         ---             ----         ---- = --
 758  1137         379             1516         5306   14
 758 | 758         379             1516
---- | ----        ---             ----
     | 379
```

In this example it may be noted that all the work is shown. The best rule to observe with regard to this matter is, to *put down* ALL *the work that it is necessary to work out on paper, and to put down* NONE *that may be done in the mind.*

Ex. 14.

Reduce to their lowest terms by successive division.

1. $\frac{84}{120}$. 2. $\frac{98}{182}$. 3. $\frac{240}{675}$. 4. $\frac{88}{110}$. 5. $\frac{175}{225}$.
6. $\frac{1400}{1848}$. 7. $\frac{1728}{1800}$. 8. $\frac{1536}{2816}$. 9. $\frac{1300}{1625}$. 10. $\frac{7392}{19656}$.
11. $\frac{18375}{28000}$. 12. $\frac{3960}{5940}$. 13. $\frac{16272}{17136}$. 14. $\frac{21330}{28944}$.

Reduce to their lowest terms by finding the G. C. M.

15. $\frac{527}{899}$. 16. $\frac{1727}{2198}$. 17. $\frac{5681}{5928}$. 18. $\frac{4494}{7169}$.
19. $\frac{12331}{17523}$. 20. $\frac{77529}{179826}$.

Fractions, like whole numbers, may be made the subjects of the four processes, Addition, Subtraction, Multiplication, and Division, and of these it will be most convenient to consider Multiplication and Division first. Let it be required to multiply together $\frac{3}{8}$ and $\frac{5}{7}$. The result must be, from what has been explained, $\frac{3}{8}$ multiplied by 5 and then divided by 7, which will be $\frac{15}{56}$. Next take the case of $\frac{3}{8}\times\frac{5}{7}\times\frac{9}{11}$. This must be the same as $\frac{15}{56}\times\frac{9}{11}$, which is $\frac{135}{616}$. The same reasoning will apply to any number of fractions, and we have therefore the following general rule for multiplication of fractions:—*Multiply the numerators together for a new numerator, and the denominators together for a new denominator.*

The general rule for division by fractions will be, as has been already explained,

Invert the divisor, and multiply.

The following observations should be attended to in the application of the above rules:—

(1) The result should always be expressed in its lowest terms. This may be done by reducing the result found by the rule to its lowest terms, as for example :

$$\frac{2}{3}\times\frac{9}{11}\times\frac{7}{12}\times\frac{22}{35}=\frac{2772}{13860}=\frac{308}{1540}=\frac{28}{140}=\frac{1}{5}$$

But a shorter method is to strike out of any of the numerators and any of the denominators a common factor as often as one can be found. This being in effect dividing both numerator and denominator of the product by the same number, does not alter its value. The example just given would then appear thus :

$$\frac{\cancel{2}}{\cancel{3}}\times\frac{\overset{3}{\cancel{9}}}{\cancel{11}}\times\frac{\cancel{7}}{\cancel{12}}\times\frac{\overset{2}{\cancel{22}}}{\underset{5}{\cancel{35}}}=\frac{1}{5}$$

Here 7 has been struck out from the numerator and from the 35 of the denominator, leaving the factor 5 ; 3 from the denominator and from the 9 of the numerator, leaving the factor 3 ; 11 from the denominator and from the 22 of the numerator, leaving the factor 2 ; 12 from the denominator, which is equivalent to the factors 2, 3, and 2 struck out of the numerator. The numerator has, therefore, been all struck out, that is, divided by itself, and the new numerator is therefore 1 ; the denominator is 5.

(2) All whole and mixed numbers must be reduced to a fractional form before the multiplication is attempted. Thus,

$$\frac{3}{25}\times 9\tfrac{3}{8}\times 6\times 2\tfrac{2}{7}\times\frac{1}{54}=\frac{\cancel{3}}{\cancel{25}}\times\frac{\overset{3}{\cancel{75}}}{\cancel{8}}\times\frac{\cancel{6}}{1}\times\frac{\overset{2}{\cancel{16}}}{7}\times\frac{1}{\cancel{54}}=\frac{2}{7}$$

(3) Although × is the sign of multiplication used with respect to fractions as well as whole numbers, yet the word 'of' is often employed between fractions to signify that they are to be multiplied together. When 'of' is used the fractions between which it stands are to be considered as one quantity, this quantity being called a *compound fraction*.

(4) Division by a fraction may be expressed in either of two ways: for instance, $\frac{3}{5}$ divided by $2\frac{1}{10}$ is $\frac{3}{5} \div 2\frac{1}{10}$, or $\dfrac{\frac{3}{5}}{2\frac{1}{10}}$. There is no difference of meaning between the two expressions, and each is equal to $\frac{\cancel{3}}{\cancel{5}} \times \dfrac{\overset{2}{\cancel{10}}}{\underset{7}{\cancel{21}}} = \frac{2}{7}$.

Ex. 15.

1. $\frac{13}{16} \times \frac{8}{9} \times \frac{15}{28} \times \frac{9}{26} \times \frac{91}{100}$.
2. $\frac{11}{12} \times \frac{24}{25} \times \frac{100}{121} \times \frac{77}{96} \times \frac{33}{56}$.
3. $\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} \times \frac{5}{6} \times \frac{6}{7}$.
4. $\frac{17}{18} \times \frac{504}{527} \times \frac{341}{560} \times \frac{16}{33}$.
5. $\frac{3}{8} \times \frac{16}{21} \times \frac{49}{50} \times \frac{75}{96} \times \frac{6}{7}$.
6. $\frac{9}{10} \times \frac{7}{8} \times \frac{11}{27} \times 2\frac{2}{7} \times 1\frac{7}{11} \times 1\frac{2}{3}$.
7. $\frac{3}{5} \times 1\frac{1}{16} \times \frac{3}{10} \times \frac{8}{21} \times 4\frac{1}{11} \times 1\frac{15}{17}$.
8. $\frac{4}{7} \times 5\frac{1}{4} \times \frac{108}{143} \times 2\frac{8}{9} \times 8\frac{1}{4}$.
9. $\frac{38}{45} \times \frac{4}{7} \times \frac{11}{15} \times 6\frac{1}{4} \times 4\frac{5}{19}$.
10. $18\frac{2}{3} \times 12\frac{6}{7} \times \frac{1}{14} \times \frac{7}{16} \times \frac{2}{15}$.
11. $\frac{100}{143} \div \frac{25}{78}$.
12. $\frac{175}{216} \div \frac{275}{84}$.
13. $1\frac{19}{45} \div 1600$.
14. $2\frac{1}{23} \div 1\frac{57}{460}$.
15. $\frac{4}{7} \div 3\frac{11}{21}$.
16. $\dfrac{8\frac{2}{9}}{12}$.
17. $\dfrac{100}{6\frac{9}{11}}$.
18. $\dfrac{28\frac{1}{2}}{7\frac{11}{12}}$.
19. $\dfrac{2\frac{2}{3} \times 1\frac{1}{8}}{4\frac{1}{6} \times \frac{3}{5}}$
20. $\dfrac{24}{2\frac{1}{3} \times \frac{11}{14} \times 3}$.

In the addition of fractions, taking first the case where the denominators are equal, the result is obviously arrived at by adding the numerators together. For example: $\frac{4}{7}$, meaning the result of dividing 4 by 7, and $\frac{6}{7}$, meaning the result of dividing $\frac{6}{7}$, it is clear that the sum of these must be the result of dividing 10 by 7. If 4 cwt. of coals had been divided among 7 men, and afterwards 6 cwt. more had been divided among them, it is certain that each man would have had as much as if 10 cwt. had been divided among them at first. Hence $\frac{4}{7} + \frac{6}{7} = \frac{10}{7} = 1\frac{3}{7}$.

When the fractions have unequal denominators, they may be reduced to others having equal denominators. Let the fractions be $\frac{4}{9}$, $\frac{7}{12}$, and $\frac{13}{18}$. Now, $\frac{4}{9}$ is equivalent to an infinite number of fractions having as denominator some multiple of 9; as, for instance, $\frac{8}{18}$, $\frac{16}{36}$, $\frac{20}{45}$, $\frac{400}{900}$, &c. And similarly, $\frac{7}{12}$ and $\frac{13}{18}$ can be replaced by an infinite number of fractions having denominators which are multiples of 12 and of 18 respectively. If these equivalent fractions,

therefore, are to have the same denominator, that denominator must be a multiple of 9, 12, and 18. Thus,

$$\tfrac{4}{9}, \tfrac{7}{12}, \tfrac{13}{18}, \text{ are equivalent to } \tfrac{80}{180}\ \tfrac{105}{180}\ \tfrac{130}{180}$$
$$\text{or to } \tfrac{400}{900}\ \tfrac{525}{900}\ \tfrac{650}{900}$$

But it is, moreover, convenient that this denominator should be as small as possible, and hence the least common multiple of 9, 12, and 18 is chosen. This we know to be 36.

To reduce $\frac{4}{9}$ to a fraction with denominator 36 we must multiply both its numerator and denominator by 4 making $\frac{16}{36}$. In the case of $\frac{7}{12}$ we must multiply by 3, making $\frac{21}{36}$; and in the case of $\frac{13}{18}$ by 2, making $\frac{26}{36}$. Therefore—

$$\tfrac{4}{9}, \tfrac{7}{12}, \tfrac{13}{18}, \text{ are equivalent to } \tfrac{16}{36}, \tfrac{21}{36}, \tfrac{26}{36}.$$

This operation is termed reducing fractions to their *Least Common Denominator*, and the general rule is the following: *Find the* L. C. M. *of the denominators, take this as the common denominator, and reduce the fractions to equivalent fractions having that denominator.*

To add fractions together, *Reduce them to their least common denominator, and add together their numerators, retaining their common denominator.* Thus—

$$\tfrac{4}{9}+\tfrac{7}{12}+\tfrac{13}{18}=\frac{16+21+26}{36}=\tfrac{63}{36}=\tfrac{7}{4}=1\tfrac{3}{4}.$$

To subtract one fraction from another, *Reduce them to their Least Common Denominator, and subtract one numerator from the other, retaining their common denominator.* Thus—

$$\tfrac{17}{56}-\tfrac{5}{24}=\frac{51-35}{168}=\tfrac{16}{168}=\tfrac{2}{21}.$$

The following remarks relate to the application of these rules:

(1) When whole and mixed numbers occur among the fractions to be added, the whole numbers and fractions should be added separately.

$$2\tfrac{1}{5} + 4 + \tfrac{3}{7} + 1\tfrac{11}{14} + \tfrac{29}{35} = 7 + \frac{14 + 30 + 55 + 58}{70} = 7 + \tfrac{157}{70} = 9\tfrac{17}{70}.$$

(2) Compound fractions should be reduced to simple fractions, and improper fractions to mixed numbers, before the application of the rule.

$$\begin{aligned}\tfrac{105}{16} + 3\tfrac{41}{80} + \tfrac{13}{15} \text{ of } \tfrac{1}{4} \text{ of } \tfrac{5}{6} + \tfrac{7}{8} &= 6\tfrac{9}{16} + 3\tfrac{41}{80} + \tfrac{13}{72} + \tfrac{7}{8}\\ &= 9 + \frac{405 + 369 + 130 + 630}{720}\\ &= 9 + \tfrac{1534}{720} = 11\tfrac{94}{720} = 11\tfrac{47}{360}.\end{aligned}$$

(3) In subtraction of fractions also, compound fractions must be reduced to simple fractions, and improper fractions should be reduced to mixed numbers before the application of the rule, and whole numbers and fractions ought to be subtracted separately. As there arise two or three cases which are best expressed in somewhat different forms, an instance of each shall be given.

Where the second fraction is less than the first.

$$4\tfrac{5}{8} - 2\tfrac{7}{12} = 2 + \frac{15-14}{24} = 2\tfrac{1}{24}.$$

Where the second fraction is greater than the first.

$$4\tfrac{5}{8} - 2\tfrac{11}{12} = 2 + \frac{15-22}{24} = 2 - \tfrac{7}{24} = 1\tfrac{17}{24}.$$

Where there is no first fraction.

$$8 - 3\tfrac{11}{48} = 5 - \tfrac{11}{48} = 4\tfrac{37}{48}.$$

In each of the two latter cases it has been necessary to consider 1 from the whole number replaced by an equivalent fraction having the common denominator. Thus in one case $2 - \frac{7}{24} = 1 + \frac{24}{24} - \frac{7}{24}$ since $\frac{24}{24}$ is the same as 1. And $1 + \frac{24}{24} - \frac{7}{24} = 1\frac{17}{24}$.

In the other case $8 - 3\frac{11}{48} = 7 + \frac{48}{48} - 3\frac{11}{48} = 4\frac{37}{48}$.

When, however, the reason of the process is once seen, there is no need to put down the intermediate step.

Every operation in Arithmetic can only be a combination of the four processes, Addition, Subtraction, Multiplication, and Division ; and the application of these to fractions has been considered. Before, however, proceeding to the next

subject in Arithmetic, there is one minor point to which it is necessary to call attention.

Let three fractions $\frac{13}{63}$, $\frac{3}{14}$, and $\frac{5}{24}$ be given, and let it be required to determine which is the greatest, and which the least. Such examples as these, involving any number of fractions, may be at once worked by reducing the fractions to a common denominator. Thus—

$$\frac{13}{63}, \frac{3}{14}, \frac{5}{24} = \frac{104,\ 108,\ 105}{504}$$

And $\frac{13}{63}$ is the least, and $\frac{3}{14}$ the greatest of the fractions.

Ex. 16.

1. $\frac{3}{4}+\frac{7}{9}+\frac{11}{12}+\frac{5}{8}+\frac{2}{3}$.
2. $\frac{13}{21}+\frac{25}{56}+\frac{17}{63}+\frac{37}{72}+\frac{43}{84}$.
3. $\frac{19}{48}+\frac{11}{25}+\frac{23}{60}+\frac{9}{16}+\frac{14}{15}$.
4. $2\frac{1}{5}+3\frac{1}{4}+4\frac{2}{11}+5+6\frac{1}{2}$.
5. $4\frac{1}{7}+10+2\frac{1}{5}+3\frac{1}{9}+\frac{17}{21}$.
6. $3\frac{2}{9}+8\frac{1}{2}+4+5\frac{1}{11}+\frac{25}{66}$.
7. $\frac{2}{7}+\frac{3}{11}+\frac{5}{33}+\frac{2}{21}+\frac{15}{77}$.
8. $5\frac{2}{3}+4\frac{1}{8}+6+2\frac{1}{10}$ of $\frac{3}{7}$ of $\frac{5}{6}+\frac{211}{20}$.
9. $\frac{2}{7}$ of $5\frac{1}{4}$ of $3\frac{1}{3}$ of $4+\frac{217}{15}+\frac{3}{10}$ of $\frac{5}{9}$ of 7.
10. $\frac{7}{8}-\frac{4}{5}$.
11. $\frac{17}{20}-\frac{2}{5}$.
12. $\frac{8}{13}-\frac{6}{11}$.
13. $\frac{5}{12}-\frac{7}{25}$.
14. $7\frac{3}{4}-3\frac{2}{5}$.
15. $10\frac{1}{2}-8\frac{3}{7}$.
16. $4\frac{9}{10}-3\frac{1}{8}$.
17. $5\frac{43}{74}-2\frac{3}{37}$.
18. $6\frac{1}{8}-4\frac{1}{2}$.
19. $17\frac{1}{5}-9\frac{7}{10}$.
20. $5\frac{1}{12}-2\frac{2}{3}$.
21. $6\frac{3}{4}-5\frac{8}{9}$.
22. $11-4\frac{2}{7}$.
23. $10-3\frac{4}{9}$.
24. $16-14\frac{2}{5}$.
25. $12\frac{1}{2}+\frac{2}{3}$ of $7\frac{4}{5}-\frac{1}{2}$ of $3\frac{1}{7}+\frac{13}{14}-\frac{127}{35}$.
26. $\frac{5}{26}$ of $11\frac{1}{7}$ of $\frac{1}{3}-\frac{4}{11}$ of $\frac{9}{14}$ of $7\frac{1}{3}+\frac{35}{3}-2\frac{2}{3}$ of 4.
27. Which is the greatest, and which the least, of the fractions $\frac{71}{105}$, $\frac{17}{25}$, $\frac{19}{28}$

CHAPTER IV.

DECIMALS; AND THE APPLICATION OF FRACTIONS AND DECIMALS TO CONCRETE QUANTITIES.

SINCE proper fractions have the numerator less than the denominator, their values must in all cases be less than one, and by means of fractions any quantity less than one may be either exactly expressed, or approximated to as nearly as we please. This is not, however, the only method of expressing such quantities. Another way of doing so is by using an extension of our ordinary system of notation.

It will be remembered that ordinary numbers are expressed by a system in which the value of the separate figures increases tenfold as we pass from figure to figure towards the left, and consequently diminishes similarly towards the right. Thus, 2436 means 2 *thousands*, 4 *hundreds*, 3 *tens*, and 6 *units*. Hitherto this system has been supposed to stop at *units*, no smaller value being given to any figures. But we might if we pleased consider it extended indefinitely, following the same law, in which case *tenths* would follow *units*, *hundredths* would come after *tenths*, and so on. Such an extension gives rise to the quantities called decimals, and a dot coming after the place of units, separating the whole numbers from the decimals, is called the decimal point. For example, 2436·1024 means 2 *thousands*, 4 *hundreds*, 3 *tens*, 6 *units*, 1 *tenth*, 0 *hundredths*, 2 *thousandths*, 4 *ten thousandths*.

It will be best to consider first the nature and properties of decimals, reserving till afterwards the explanation of their relations to fractions. 28·75 means 2 *tens*, 8 *units*, 7 *tenths*, 5 *hundredths*. Let the same figures be retained, and let the

decimal point be moved one place to the left. Then 2·875 means 2 *units*, 8 *tenths*, 7 *hundredths*, 5 *thousandths*. Every figure has thus been made to represent one tenth of its former value, and the whole quantity has therefore been divided by ten. Hence $28 \cdot 75 \div 10 = 2 \cdot 875$. Had the decimal point been moved two places to the left, the expression would have been changed to ·2875, or 2 *tenths*, 8 *hundredths*, 7 *thousandths*, 5 *ten thousandths*. Here each figure has been made to represent one hundredth of its former value, and $28 \cdot 75 \div 100 = \cdot 2875$. Similarly, had the decimal point been moved one place to the right, the quantity would have been multiplied by ten, if two places, by a hundred: $28 \cdot 75 \times 10 = 287 \cdot 5$; $28 \cdot 75 \times 100 = 2875$. The decimal point may be moved any number of places to either the left or right by the employment of ciphers. Thus, in moving it 8 places to the left, and 7 places to the right, we should have respectively—

$$28 \cdot 75 \div 100000000 = \cdot 0000002875$$
$$\text{and } 28 \cdot 75 \times 10000000 = 287500000.$$

As it is very important to understand the effect produced by the change of place of the decimal point, some examples are given to illustrate it :—

$730 \cdot 29 \times 1000 = 730290.$	$730 \cdot 29 \div 1000 = \cdot 73029.$
$730 \cdot 29 \times 10 = 7302 \cdot 9.$	$730 \cdot 29 \div 10 = 73 \cdot 029.$
$730 \cdot 29 \times 100000 = 73029000.$	$730 \cdot 29 \div 100000 = \cdot 0073029.$

The following particulars must be borne in mind :—

1. After whole numbers, or *integers*, as they are called, we omit the decimal points. Thus, 16 means 16·; 80000 means 80000·. If this decimal point be expressed, any number of ciphers afterwards added make no difference. Thus 16 is the same as 16·000; 80000 is the same as 80000·00000.

2. With respect to figures to the *left* of the decimal point, ciphers *before the first* significant figure (that is, figure other than 0), make no difference, ciphers *after the last* significant figure increase the value.

For instance, 000819· is the same as 819·,
but $819000 \cdot = 819 \cdot \times 1000$.

3. With respect to figures to the *right* of the decimal point, ciphers *after the last* significant figure make no difference, ciphers *before the first* significant figure diminish the value.

For instance, ·020078000 is the same as ·020078,
but ·000020078 = ·020078 ÷ 1000.

The advantage of this mode of expressing quantities less than 1 is, that decimals, following the same law as ordinary numbers, may be added, subtracted, multiplied, and divided, precisely in the same manner, the only additional difficulty being to determine the place of units in the result, or, in other words, where to put the decimal point. In Addition and Subtraction it is sufficient to take care that units are added to units, tens to tens, and the like; and this is ensured by following the rule, '*Keep the decimal points one under another.*' The following examples will illustrate this.

$$\begin{array}{r@{\cdot}l}
24 & 371 \\
15 & \\
 & 036 \\
8 & 4 \\
2 & 0974 \\
\hline
49 & 9044
\end{array}
\qquad
\begin{array}{r@{\cdot}l}
257000 & \\
307 & 00008 \\
11 & \\
 & 96 \\
\hline
257318 & 96008
\end{array}
\qquad
\begin{array}{r@{\cdot}l}
234 & 1 \\
207 & 4376 \\
\hline
26 & 6624
\end{array}
\qquad
\begin{array}{r@{\cdot}l}
179 & 2783275 \\
111 & 487 \\
\hline
67 & 7913275
\end{array}$$

Ex. 17.

Add together the decimals,

1. 29·705, ·089, 3·26, 41·7, 2·845.
2. 31·826, 3·471, ·004, 45, ·6.
3. 64·27, 1·1, 23, 17·12, 8·8.
4. 82·537, 2000, 1·354, ·006, 13.
5. 1000·0001, 100·001, 10·01, 1·1, 2·2, 3·3, 20·02, 3000·0003.

Subtract

6. 4·26 from 8·137.
7. 17·294 from 192·3.
8. ·0076 from 517.
9. 13·794 from 186·257.
10. 239·385 from 253·812.

In multiplication and division, the plan adopted is, in the first place, to consider the decimals as whole numbers, and

afterwards correct the error made by so regarding them. A few examples will show how this is done, and lead to a general rule. Let it be required to find the value of 17·28 × 13·2. Were these whole numbers, the product would be 228096. But, to make them whole numbers, the decimal points must have been supposed moved towards the right two places and one place respectively, thus, in fact, multiplying them by 100 and by 10. To make up for this, let the decimal point in the product be moved three places towards the left, thus dividing by 1000, and making the result 228·096.

Again, to find the value of 4·297 × ·0012.

The result, if they were whole numbers, would be 51564, but that supposes the decimal points to have been moved three and four places towards the right. Hence, in the product, the decimal point must be moved seven places towards the left, and the result is ·0051564.

Next, in division, the errors arising from supposing the divisor and dividend whole numbers will act in opposite directions. Taking the divisor as a whole number will in general increase it, and make the quotient too small. Taking the dividend as a whole number will make the quotient too large. The decimal point in the quotient must therefore be moved so as to make up for the excess of the larger error over the smaller. Thus, in dividing 43·6508 by 28·4, if they were whole numbers, the result would be 1537. But this supposition multiplies the dividend, and hence multiplies the quotient by 10000; while it also multiplies the divisor, and hence divides the quotient by 10. Altogether, therefore, the quotient has been multiplied by 1000, and, to make up for this, the decimal point must be moved three places towards the left, making the result 1·537.

Again, to find ·00436508 ÷ 284000.

Here three ciphers must be placed after the dividend, to enable the division to take place at all. To make the quantities whole numbers, the dividend must then be multiplied

by 100000000000, while the divisor is unchanged. Hence the result must be 1537 with the decimal point moved eleven places, or ·00000001537.

Again, to find 43650·8 by ·0000284.

Here the dividend is supposed multiplied by 10, and the divisor by 10000000; the quotient will therefore be 1000000 times too small. The decimal point must be moved six places to the right, and thus, in place of 1537, the quotient becomes 1537000000.

Lastly, 436·508 by ·284.

Here the errors exactly counteract one another, and the quotient is 1537.

From the consideration of these particular instances, there result the following general rules:—

MULTIPLICATION OF DECIMALS.—*Multiply as in whole numbers, and move the decimal point to the left as many places as there are decimal places in the multiplier and multiplicand together.*

DIVISION OF DECIMALS.—*Divide as in whole numbers, adding ciphers to the dividend if necessary; and move the decimal point to the right if the divisor have more decimal places than the dividend, and to the left if the dividend have more than the divisor, in either case moving it as many places as make up the difference.* (Note I.)

In division of decimals it is often easy to assign the position of the decimal point at once, without counting the decimal places. Thus 480·70302÷79·851 gives in the quotient the figures 6, 0, 2; and as 480÷79 is clearly less than 10, the quotient must obviously be 6·02.

Ex. 18.

Find the values of

1. 11·2 × 3·4.	2. 3·75 × 42.
3. 2·39 × ·037.	4. 28·015 × 4·3.
5. ·08 × 400.	6. ·5679 × ·00673.
7. 48·76 × ·138.	8. 53·81 × 2700.
9. 12479 × ·000061.	10. 78·125 × 10·24.
11. ·03 × 1·6 × 25 × 7 × ·001.	12. 1·7 × 110 × 13 × ·006 × 24.

13. 1800 × ·003 × 60 × 10 × ·0045.
14. ·1 × ·2 × ·3 × ·4 × ·5 × ·6 × ·7 × ·8 × ·9.

15. ·00128 ÷ ·8.	16. 156·25 ÷ ·025.
17. 19·008 ÷ ·0176.	18. 23·25 ÷ 372.
19. 233·937 ÷ 2·78.	20. 34·04798 ÷ ·08416.
21. 22·7088 ÷ 3800.	22. 96683·4 ÷ ·4112.
23. 39·02286 ÷ 10425.	24. 714535·25 ÷ 69·52.
25. 2022·91 ÷ ·00043.	26. 346·7233 ÷ ·0039.

If it be remembered, as explained in Chapter I., that the origin of fractions was owing to our being obliged to stop the process of ordinary division of whole numbers when we had arrived at the place of units in the dividend, it is evident that when the notation is extended, so that we are not obliged to stop, the quantity formerly expressed by a fraction may be expressed in some other form. As an example, take the fraction $\frac{7}{8}$. This means the result of dividing 7 by 8, a process in Chapter I. deemed impossible. But let 7 be written 7·000, which means 7 *units*, 0 *tenths*, 0 *hundredths*, 0 *thousandths*, and let it be remembered that the law of local value of the decimals is exactly the same as that of whole numbers. Hence the process of division will be the same also, and, dividing as in whole numbers, we have the quotient ·875. This expression means, as has been explained, 8 *tenths*, 7 *hundredths*, 5 *thousandths*; and $\frac{8}{10}+\frac{7}{100}+\frac{5}{1000}=\frac{875}{1000}=\frac{7}{8}$. It will be seen, therefore, that $\frac{7}{8}$ and ·875 are different expressions for the same quantity. Next take the fraction $\frac{7}{80}$. Here, again, let ciphers be added to 7, and separated from it by the decimal point, and let it then be divided by 80. The quotient will evidently contain the same digits, 8, 7, and 5, as before; but inasmuch as 80 is not contained in 7 or in 70, there will be no *units* digit or *tenths* digit, and since 80 is contained in 700, there will be a *hundredths* digit, namely 8. The quotient will consequently be 8 *hundredths*, 7 *thousandths*, and 5 *ten thousandths*, or ·0875. When, therefore, we employ all the additional power of division that the extension of our notation to decimals furnishes, we can divide a less number by a greater,

and thus express the value of a fraction in another form. The rule for reducing a fraction to a decimal is, *Divide the numerator by the denominator.*

$$\frac{7}{32}=\cdot21875 \quad \frac{239}{625}=\cdot3824 \quad \frac{17}{2000}=\cdot0085= \quad \frac{3}{800}=\cdot00375.$$

There are two distinct cases of this division of the numerator by the denominator, one where the division terminates without any remainder being left, the other where the division would continue for ever. The four examples given above are instances of a complete division, and the resulting decimals are *finite* or *terminating.* If the fraction is in its lowest terms, the condition necessary to insure a complete division is that the denominator shall not contain any other factors than 2 or 5. For every cipher added to the dividend has the effect of increasing it (when considered as a whole number) ten times, and thereby introducing the factors 2 and 5. By adding on a sufficient number of ciphers, these particular factors may be introduced as often as we please, but no other factors whatever can be brought in. Hence a fraction in its lowest terms, whose denominator contains any other factor than 2 or 5, must give rise to a decimal which does not terminate. As an instance take the fraction $\frac{173}{259}$. Dividing 173 by 259, we have a quotient ·667953, and after that a remainder 173, which is the same as the original dividend. The same set of numbers will consequently be repeated over and over again in the quotient, so that $\frac{173}{259}=$ ·667953667953667953 &c. for ever. As a short way of expressing this quantity, it is written $\cdot\dot{6}6795\dot{3}$, a dot being placed above the first and last of the recurring figures. Such a decimal is called a *recurring* or *circulating decimal.*

```
259)173·0(·667953
    155 4
    -----
     17 60
     15 54
     -----
      2 060
      1 813
      -----
        2470
        2331
        ----
         1390
         1295
         ----
          950
          777
          ---
          173
```

Again, let $\frac{135}{328}$ be the fraction. Then, after dividing until we have in the quotient $\cdot 41158536$, the remainder is 192; and this is identical with a former remainder, which produced the figures 58536 in the quotient. These figures consequently recur, and $\frac{135}{328} = \cdot 411\dot{5}853\dot{6}$. A decimal in which some of the figures recur while others do not is sometimes called a *mixed circulating decimal.*

$$
\begin{array}{l}
328)135\cdot 0(\cdot 41158536 \\
\quad\;\; 131\cdot 2 \\
\quad\;\;\;\; 3\;80 \\
\quad\;\;\;\; 3\;28 \\
\quad\;\;\;\;\; 520 \\
\quad\;\;\;\;\; 328 \\
\quad\;\;\;\;\; 1920 \\
\quad\;\;\;\;\; 1640 \\
\quad\;\;\;\;\;\; 2800 \\
\quad\;\;\;\;\;\; 2624 \\
\quad\;\;\;\;\;\;\; 1760 \\
\quad\;\;\;\;\;\;\; 1640 \\
\quad\;\;\;\;\;\;\;\; 1200 \\
\quad\;\;\;\;\;\;\;\;\; 984 \\
\quad\;\;\;\;\;\;\;\;\; 2160 \\
\quad\;\;\;\;\;\;\;\;\; 1968 \\
\quad\;\;\;\;\;\;\;\;\;\;\; 192
\end{array}
$$

Certain decimals should be remembered, because they are the equivalents of fractions which are in constant use, and by remembering them the time of converting them from one form to the other is saved. The most important are $\frac{1}{2} = \cdot 5$, $\frac{1}{4} = \cdot 25$, $\frac{3}{4} = \cdot 75$, $\frac{1}{3} = \cdot\dot{3}$, $\frac{2}{3} = \cdot\dot{6}$, $\frac{1}{9} = \cdot\dot{1}$. The following are also useful:— $\frac{1}{6} = \cdot 1\dot{6}$, $\frac{1}{8} = \cdot 125$, $\frac{1}{16} = \cdot 0625$, $\frac{1}{5} = \cdot 2$, $\frac{2}{5} = \cdot 4$, $\frac{3}{5} = \cdot 6$, $\frac{4}{5} = \cdot 8$. A curious relation exists between the decimals respectively equivalent to the fractions $\frac{1}{7}$, $\frac{2}{7}$, $\frac{3}{7}$, $\frac{4}{7}$, $\frac{5}{7}$, $\frac{6}{7}$. They all consist of a recurring period of six figures; the figures in each are the same; the same figures precede and follow any given figure in each. The only difference between them is that the recurring cycle is commenced at a different point. So that if the figures were placed at equal distances round a circle, and a period made by commencing at any one and taking them in order all round, the periods expressing the above six fractions would be formed. Thus:

$\frac{1}{7} = \cdot\dot{1}4285\dot{7}$
$\frac{2}{7} = \cdot\dot{2}8571\dot{4}$
$\frac{3}{7} = \cdot\dot{4}2857\dot{1}$
$\frac{4}{7} = \cdot\dot{5}7142\dot{8}$
$\frac{5}{7} = \cdot\dot{7}1428\dot{5}$
$\frac{6}{7} = \cdot\dot{8}5714\dot{2}$

7 1
5 4
8 2

It is not on account of any special peculiarity in the

number 7 that this relation exists. The same law holds good for fractions with denominators 17, 19, 23, 29, and in general wherever the denominator is 1 more than the number of figures in the circulating period. The number 7, however, gives the first instance of this kind, and, moreover, is a number which often occurs in calculation, since, for example, 7 days make a week, and 28 lbs. make a quarter. And therefore it is well to notice how the law applies to this particular case.

Ex. 19.

Reduce the following fractions to decimals:—

1. $\frac{13}{32}$.	2. $\frac{39}{4000}$.	3. $\frac{53}{1600}$.	4. $\frac{507}{625}$.
5. $\frac{23}{200}$.	6. $\frac{47}{2560}$.	7. $16\frac{9}{40}$.	8. $11\frac{67}{80}$.
9. $14\frac{79}{4000}$.	10. $\frac{11}{18}$.	11. $\frac{14}{27}$.	12. $\frac{107}{296}$.
13. $\frac{419}{7000}$.	14. $\frac{311}{328}$.	15. $25\frac{19}{101}$.	16. $37\frac{25}{48}$.
17. $4\frac{29}{440}$.	18. $\frac{149}{407}$.	19. $41\frac{195}{292}$.	20. $\frac{3}{260}$.

Having thus shown how fractions may be reduced to decimals, it remains to explain how decimals may be reduced to fractions; and here there are two cases to consider—first, where the decimal terminates; secondly, where it does not terminate—the first being excessively easy, and the second a little more difficult.

In the case of a terminating decimal, it is only necessary to multiply it by such a power of 10 as will make it a whole number, and divide it back again, by placing the same power of 10 in the denominator of a fraction of which the whole number is the numerator. Thus :—

$$1\cdot874=\frac{1874}{1000} \text{ or } 1\frac{874}{1000} \text{ or } 1\frac{437}{500} \qquad \cdot0007263=\frac{7263}{10000000}$$

This reduction of decimals to fractions leads to another light in which to regard a decimal, namely, as being *a fraction with the denominator left out, that denominator being* 1 *followed by as many ciphers as there are figures after the decimal point.*

The rule for reducing circulating decimals to fractions

is a little complicated, and the method of arriving at it is not easy to explain. The best way will perhaps be to take an instance, and, by working this out from first principles, arrive at a rule. It must be first premised, as will be found by actual division, that

$$\tfrac{1}{9} = \cdot 11111111 \text{ \&c. or } \cdot\dot{1};$$
$$\tfrac{1}{99} = \cdot 01010101 \text{ \&c. or } \cdot\dot{0}\dot{1};$$
$$\tfrac{1}{999} = \cdot 001001001 \text{ \&c. or } \cdot\dot{0}0\dot{1};$$
$$\tfrac{1}{9999} = 00010001 \text{ \&c. or } \dot{0}00\dot{1}; \text{ and so on.}$$

Now, to find the fraction equivalent to $\cdot 0597\dot{9}7\dot{2}$, this decimal is evidently equal to the sum of $\cdot 0597$ and $\cdot 000\dot{9}7\dot{2}$.

Now $\quad \cdot\dot{9}7\dot{2}$ must be equal to $972 \times \cdot 001001001$ &c. or $\dfrac{972}{999}$

Hence $\quad \cdot 0000\dot{9}7\dot{2} = \dfrac{1}{10000} \times \dfrac{972}{999} = \dfrac{972}{9990000}$; and $\cdot 0597 = \dfrac{597}{10000}$

Therefore $\cdot 0597\dot{9}7\dot{2} = \dfrac{597}{10000} + \dfrac{972}{9990000}$

$$\cdot 0597\dot{9}7\dot{2} = \frac{(597 \times 999) + 972}{9990000} = \frac{597000 - 597 + 972}{9990000}$$
$$= \frac{597972 - 597}{9990000}$$

The numerator is the difference between the decimal considered as a whole number and the part not circulating, and the denominator has as many nines as there are circulating figures, followed by as many ciphers as there are figures non-circulating.

The fraction must of course be expressed in its lowest terms, thus :—

$$\frac{597972 - 597}{9990000} = \frac{597375}{9990000} = \frac{23895}{399600} = \frac{4779}{79920} = \frac{531}{8880} = \frac{177}{2960}$$

Again, $6\cdot 69602\dot{2}\dot{7} = 6 + \dfrac{6960227 - 69602}{9900000} = 6\tfrac{6890625}{9900000} = 6\tfrac{55125}{79200} = 6\tfrac{2205}{3168}$ $= 6\tfrac{245}{352}$.

$$\cdot 49\dot{3}2\dot{4} = \frac{49324 - 49}{99900} = \frac{49275}{99900} = \frac{1971}{3996} = \frac{219}{444} = \frac{73}{148}$$

From this rule it follows that $\cdot\dot{9}=\frac{9}{9}=1$. This is a good illustration of a subject with which the student will become more and more familiar as he advances in mathematics, namely, the sum of an infinite series. $\cdot\dot{9}$ is merely an abbreviation for $\cdot 9999$, &c., continued for ever, and the farther this series is continued, the nearer it approaches unity. Thus $\cdot 9$ is $\frac{1}{10}$ less than unity, $\cdot 99$ is $\frac{1}{100}$ less, $\cdot 99999$ is $\frac{1}{100000}$ less, and so on. Also, by going on with the series sufficiently far, we may make its value as near to unity as we please. These two conditions constitute what in mathematics is termed a 'limit,' and hence 1 is the limit of the sum of the series $\cdot 9999$, &c., continued to infinity.

It may be shown, by Algebra, that all circulating decimals whatever are series continued to infinity, and that the fractions equivalent to them are the 'limits' of the sums of these series.

Ex. 20.

Reduce the following decimals to fractions.

1. $\cdot 21875$. 2. $\cdot 00176$. 3. $5\cdot 072$.
4. $4\cdot 00237312$. 5. $\cdot 4296875$. 6. $3\cdot 9375$.
7. $\cdot 08395$. 8. $\cdot 7953125$.

Prove, by reducing to decimals, the truth of the following results:—

9. $\frac{1}{2}+\frac{5}{8}+\frac{3}{4}=1\frac{7}{8}$. 10. $\frac{5}{16}\times\frac{7}{25}=\frac{7}{80}$. 11. $3\frac{5}{8}-2\frac{3}{4}=\frac{7}{8}$.

Reduce the following circulating decimals to fractions.

12. $\cdot 5\dot{9}\dot{0}$. 13. $12\cdot 458\dot{3}$. 14. $8\cdot 2\dot{6}$.
15. $7\cdot\dot{1}3\dot{5}$. 16. $\cdot 42\dot{6}$. 17. $13\cdot 0795\dot{4}$.
18. $\cdot 1\dot{5}64\dot{3}$. 19. $10\cdot 49\dot{8}\dot{1}$. 20. $\cdot 03\dot{1}4\dot{8}$.
21. $\cdot\dot{5}129\dot{1}$. 22. $13\cdot 05\dot{6}65271\dot{9}$. 23. $21\cdot 6105\dot{7}6923\dot{0}$.
24. $\cdot 36\dot{9}756\dot{0}$.

Since circulating decimals are thus merely abbreviated expressions for endless series, it follows that they cannot in general be added to, subtracted from, multiplied or divided by one another, without being previously reduced to fractions. Thus:—

$$2\cdot1\dot{3}\dot{6}+\cdot4\dot{8}8\dot{2}=2\tfrac{135}{990}+\frac{4878}{9990}=2\tfrac{3}{22}+\frac{271}{555}=2+\frac{1665+5962}{12210}=2+\frac{7627}{12210}=2\cdot6\dot{2}4651\dot{9}$$

$$3\cdot\dot{5}\dot{7}-2\cdot\dot{8}=3\tfrac{57}{99}-2\tfrac{8}{9}=1+\frac{57-88}{99}=1-\frac{31}{99}=\frac{68}{99}=\cdot\dot{6}\dot{8}$$

$$\cdot6\dot{7}\dot{2}\times\cdot3\dot{3}7\dot{8}=\frac{666}{990}\times\frac{3375}{9990}=\frac{5}{22}=\cdot2\dot{2}\dot{7}$$

$$\cdot1\dot{9}\div\cdot00\dot{4}\dot{5}=\frac{18}{90}\div\frac{45}{9900}=\frac{1}{5}\times\frac{220}{1}=44$$

This method of operating upon circulating decimals by previously reducing them to fractions is always applicable, and always leads to exact results, but it is often long and troublesome. In the cases of addition and subtraction, it is often enough to write the decimals down to a sufficient number of places to enable us to detect the circulating period in their sum, or at any rate to approximate to the actual result as near as may be thought necessary.

Thus, again taking the instances $2\cdot1\dot{3}\dot{6}+\cdot4\dot{8}8\dot{2}$ and $3\cdot\dot{5}\dot{7}-2\cdot\dot{8}$.

2·136363636363 &c.	3·57575757 &c.
·488288288288 &c.	2·88888888 &c.
·624651924651 &c.	·68686869
or $\cdot6\dot{2}4651\dot{9}$	or $\cdot\dot{6}\dot{8}$

In multiplication and division, it is always best to follow the general rule. Sometimes work may be shortened by reducing the circulating part of the decimal to a fraction, still retaining it in its proper place in the whole decimal.

Thus $\cdot003742\dot{3}\dot{6}\times42=\cdot003742\tfrac{4}{11}\times42=\cdot157179\tfrac{3}{11}=\cdot157179\dot{2}\dot{7}$.

It has been explained, that terminating decimals are equivalent to fractions having in their denominators the factors 2 and 5 only. Fractions having other factors in their denominators are equivalent to circulating decimals, and the number of non-circulating figures is the same as that of the highest power of either 2 or 5 which is contained in the denominator. The number of circulating figures is the same as the least number of nines placed together so as to form a number which the remaining factor of the

denominator will divide without a remainder. Thus, $\frac{21}{1480} = \frac{21}{2^3 \times 5 \times 37}$, and 37 is a factor of 999. Therefore, in the equivalent decimal there must be three non-circulating and three circulating figures. Again, $\frac{58}{1825} = \frac{58}{5^2 \times 73}$, and 73 is a factor of 99999999. Therefore, in the equivalent decimal there must be two non-circulating and eight circulating figures.

The number of nines that must be taken to form a number containing any given *prime* factor must be, in the first place, not greater than the factor *minus* one; and in the second place, if it be less than this value, it must be a factor of it. For example, the prime factor 647 requires 646 nines, and consequently gives rise to a circulating period of 646 figures. Again, the prime factor 617 gives rise to a period of 88 figures, and 88 is a factor of 616. Neither of the above laws applies to a composite factor, but whenever the periods corresponding to its prime factors are known, that corresponding to the composite factor itself may be easily deduced. Thus, $\frac{95}{7474} = \frac{95}{2 \times 37 \times 101}$, and 37 is a factor of 999, and 101 of 9999. The L.C.M. of 999 and 9999 is 9 repeated twelve times, and hence the circulating period must contain twelve figures, while the non-circulating part contains one. The following is a table of the prime factors of numbers formed by 9 repeated any number of times up to 16:—

9 once $=3^2$, twice $=3^2 \times 11$. 3 times $=3^3 \times 37$.
4 times $=3^2 \times 11 \times 101$. 5 times $=3^2 \times 41 \times 271$.
6 times $=3^3 \times 7 \times 11 \times 13 \times 37$. 7 times $=3^2 \times 239 \times 4649$.
8 times $=3^2 \times 11 \times 73 \times 101 \times 137$. 9 times $=3^4 \times 37 \times 333667$.
10 times $=3^2 \times 11 \times 41 \times 271 \times 9091$. 11 times $=3^2 \times 21649 \times 513239$.
12 times $=3^3 \times 7 \times 11 \times 13 \times 37 \times 101 \times 9901$.
13 times $=3^2 \times 53 \times 79 \times 265371653$.
14 times $=3^2 \times 11 \times 239 \times 4649 \times 909091$.
15 times $=3^3 \times 31 \times 37 \times 41 \times 271 \times 2906161$.
16 times $=3^2 \times 11 \times 17 \times 73 \times 101 \times 137 \times 5882353$.

From an inspection of this table it will be seen that there are but few factors giving rise to circulating decimals of moderately extended periods. It would require great labour to extend this table farther, and would, indeed, soon become impossible.

A few instances of factors corresponding to longer periods than those already given are:—

Period	Factors	Period	Factors
18	19	28	29, 281
20	3541	29	3191
21	43, 1933	30	211, 241, 2161
22	23, 121, 4093, 8779	31	2791
26	859	32	353, 449, 641, 1409, 69857
27	243		

As a partial test of the accuracy of the work of reducing fractions to circulating decimals, it may be noticed that all factors of the repeated nines which are not factors of the denominator of the fraction must be factors of the repeating period of the decimal. Thus we know that $\frac{197}{2340}$ would give a period of 6 figures, and since 999999 is $3^3 \times 7 \times 11 \times 13 \times 37$, and 2340 is $2^2 \times 3^2 \times 5 \times 13$, the period of the decimal, whatever it may be, must be divisible by 3, 7, 11, and 37: in fact, $\frac{197}{2340} = \cdot 08\dot{4}1880\dot{3}$, and $418803 = 3 \times 7^3 \times 11 \times 37$.

The most useful applications of this principle are such as refer to the factors 3, 9, and 11; the tests of divisibility by them being very simple.

Although, theoretically, every fraction may be exactly represented by some circulating decimal, still in most instances the length of the circulating period is such as to render it far too cumbrous to work with. When this is the case, and it is necessary to replace the fraction by a decimal, we must be contented with an approximation to the truth, and only require the decimal to be correct to a certain number of decimal places. The student will find as he advances in mathematics numerous instances where decimals occur which have not been originated by the reduc-

tion of fractions, but have been obtained by other means, and these decimals are in general non-circulating and non-terminating. The ratio of the circumference to the diameter of a circle, most cases of square roots, and the numerical quantities called 'logarithms,' are instances of this kind. In these a greater or less approach to accuracy may be made by calculating them to twelve or to six places of decimals, but complete accuracy is impossible by this or any other mode of notation. Thus, the ratio of the circumference to the diameter of a circle is 3·14159 true to 5 places of decimals, is 3·1415926536 true to 10 places, is 3·14159265358979323846 true to 20 places, and so on. When decimals of this kind are added to, or subtracted from one another, it is usual, if practicable, to carry on the decimal to one or two places more than the number ultimately required to be correct, in order to ensure the accuracy of the last figure.

For example find the value of $1\frac{493}{707} - \{\cdot 18\dot{7}\dot{5} + \frac{2}{3} + \cdot 18\dot{7}3\dot{8}\}$ correct to six places of decimals.

$$
\begin{array}{rl}
1\frac{493}{707} = 1\cdot 69731259 \text{ \&c.} & \\
1\cdot 04162981 & \\
\hline
\cdot 65568278 &
\end{array}
\qquad\qquad
\begin{array}{r}
\cdot 18757576 \\
\cdot 66666666 \\
\cdot 18738739 \\
\hline
1\cdot 04162981
\end{array}
$$

Therefore the result is ·655683 correct to six places of decimals. This is nearer the truth than ·655682, inasmuch as the figure in the next place of decimals is greater than 5.

The exact answer to the example is $\cdot 65\dot{5}6827767718\dot{8}$.

Ex. 21.

Find the values of

1. $\cdot 3541\dot{6} + \cdot 3125$. 2. $\cdot 43\dot{1}\dot{8} + \cdot\dot{4}55\dot{4}$.
3. $2\cdot 0338541\dot{6} + \cdot\dot{8}3\dot{7}$.
4. $3\cdot 458\dot{3} + 1\cdot 3\dot{1}\dot{8} + 5\cdot 75 + \cdot\dot{1}38\dot{6}$.
5. $\cdot\dot{8}5\dot{1} - \cdot 25 + \cdot 1\dot{6} - \cdot\dot{3}5\dot{1} + \cdot 75 - \cdot 91\dot{6}$.
6. $3\cdot 125 - 2\cdot 08\dot{3} + 2\cdot 458\dot{3} - \cdot 26041\dot{6} + \cdot 5 - \cdot 15\dot{0}7\dot{4}$.
7. $\cdot\dot{9}2\dot{5} \times \cdot\dot{2}67\dot{3}$. 8. $\cdot 2\dot{1}15384\dot{6} \times \cdot 443\dot{1}\dot{8}$.
9. $\cdot 340\dot{9} \times \cdot\dot{8}1\dot{4}$. 10. $71\cdot 8\dot{6} \times \cdot 0\dot{1}1688\dot{3}$.

11. $\cdot\dot{8}536\dot{5}\times\cdot1\dot{3}3579\dot{9}$. 12. $2\cdot\dot{9}7\dot{2}\times\cdot3\dot{3}\dot{6}$.

13. $2\cdot5208\dot{3}\div\cdot11458\dot{3}$. 14. $\cdot011\dot{3}\dot{6}\div10\cdot41\dot{6}$.

15. $\cdot\dot{2}\dot{7}\times4\cdot4\times2\cdot\dot{7}\times6\cdot1\dot{6}\times\cdot0\dot{7}5\dot{6}\times\cdot\dot{4}2857\dot{1}$.

16. $\cdot\dot{4}75\dot{2}\times\cdot\dot{8}\times15\cdot78125\times\cdot3\dot{6}\times\cdot07$.

Find how many non-circulating and how many circulating figures there are in the decimals equivalent to the fractions

17. $\frac{64}{185}$; $\frac{718}{5125}$. 18. $\frac{107}{584}$; $\frac{197669}{205000}$.

19. $\frac{708}{902}$; $\frac{18973}{20705}$. 20. $\frac{5761}{8424}$; $\frac{7564}{38909}$.

Find the values (correct to 6 places of decimals) of

21. $4\cdot2\dot{3}7\dot{1}+\cdot3685+5\frac{4}{9}+\frac{23}{7380}+\cdot\dot{1}46\dot{8}$.

22. $6\frac{2}{11}+3\cdot00\dot{7}3\dot{2}+\cdot\dot{1}46\dot{7}+\frac{1}{4}+7\frac{1}{3}$.

23. $7\frac{11}{28}-5\frac{197}{11000}$.

The cases of the application of fractions and decimals to concrete quantities may be resolved into four main classes, which will be considered separately.

I. To find the value of a fraction of a quantity is precisely the same thing as to multiply the quantity by the fraction. Thus; find the value of $\frac{37}{64}$ of £1 4*s.*, and multiply £1 4*s.* by $\frac{37}{64}$ are merely different ways of expressing the same question. The multiplication by a fraction means in this, as in all other cases, multiplication by the numerator and division by the denominator, but in the actual working out a little judgment is required. The quantity may be expressed in some one denomination, or in several, and sometimes the one expression is the more convenient, sometimes the other. Thus the question just proposed may be worked out in either of the following ways:—

First Method.

	£	s.	d.
	1	4	0
			3
	3	12	0
			12
	43	4	0
	1	4	0
8)	44	8	0
8)	5	11	0
		13	$10\frac{1}{2}$

Second Method.

$$\overset{s.}{\frac{37}{64}}\times\overset{s.}{\frac{24}{1}}=\overset{s.}{\frac{111}{8}}=13\tfrac{7}{8}=13s.\ 10\tfrac{1}{2}d.$$

Here the advantage is clearly on the side of reducing to one denomination.

Next take the instance £67 16*s.* $4\frac{1}{4}$*d.* $\times \frac{6}{7}$:—

First Method.

$$\begin{array}{rrr} £67 & 16 & 4\frac{1}{4} \\ & & 6 \\ \hline 7)406 & 18 & 1\frac{1}{2} \\ \hline 58 & 2 & 7\frac{1}{14} \end{array}$$

Second Method.

$$\frac{6}{7} \times \overset{s.}{1356\tfrac{17}{48}} = \frac{6}{7} \times \overset{s.}{\frac{65105}{48}} = \frac{65105}{56} = 1162s.\ 7\tfrac{1}{14}d. = £58\ 2s.\ 7\tfrac{1}{14}d.$$

$$\begin{array}{r} 1356\frac{17}{48} \\ 48 \\ \hline 10865 \\ 5424 \\ \hline 65105 \end{array} \qquad \begin{array}{rr} 8)65105 & \\ \hline 7)\ 8138 & 1\frac{1}{8} \\ \hline 1162 & 7\frac{1}{14} \end{array}$$

Here the advantage is clearly on the side of not reducing to one denomination. Division by a fraction is the same as multiplication by the fraction inverted, thus:—

$$£7\ 8s.\ 2d. \div 1\tfrac{3}{11} = £7\ 8s.\ 2d. \times \frac{11}{14} = \overset{s.}{148\tfrac{1}{6}} \times \frac{11}{14} = \overset{s.}{\frac{889}{6}} \times \frac{11}{14} = \overset{s.}{\frac{1397}{12}} = 116\tfrac{5}{12}s. = £5\ 16s.\ 5d.$$

Additional Examples. $3\frac{5}{18}$ of 7*s.* 6*d.* $= \frac{59}{18} \times \overset{s.}{\frac{15}{2}} = \overset{s.}{\frac{295}{12}} = 24\frac{7}{12}s. =$ £1 4*s.* 7*d.*

$\frac{13}{32}$ of 1 lb. troy $= \frac{13}{32} \times \overset{oz.}{\frac{12}{1}} = \overset{oz.}{\frac{39}{8}} = 4\frac{7}{8}$ oz. = 4 oz. $17\frac{1}{2}$ dwt. = 4 oz. 17 dwt. 12 gr.

Note that in this example the work is not shown by which $\frac{7}{8}$ oz. is found to be equal to $17\frac{1}{2}$ dwt. For we reduce ounces to pennyweights by multiplication by 20, and that will in this case be effected by dividing the denominator by 4, and multiplying the numerator by 5, making $\frac{35}{2}$ or $17\frac{1}{2}$, and in these operations there is no work which requires to be done on paper.

$6\frac{2}{11}$ of 2 miles 3 furlongs, $\overset{m.}{\frac{68}{11}} \times \overset{m.}{\frac{19}{8}} = \overset{m.}{\frac{323}{22}} = 14\frac{15}{22} = \overset{m.\ f.\ yd.}{14\ 5\ 100.}$

Ex. 22.

Find the values of

1. $\frac{3}{8}$ of 5*s.*; $\frac{5}{9}$ of a guinea; $\frac{17}{32}$ of £1.
2. $\frac{2}{9}$ of £5; $\frac{3}{11}$ of £9 7*s.* $10\frac{3}{4}$*d.*; 17*s.* $7\frac{1}{5}$*d.* $\times 4\frac{3}{8}$.
3. $\frac{13}{84}$ of 2 tons 5 cwt.; 1 cwt. 1 qr. $\times 23\frac{15}{32}$.

4. $3\frac{1}{18}$ of 2 lbs. troy; 11 lbs. 4 oz. 9 dwts. 19 gr. $\div 2\frac{11}{17}$.
5. £16 12*s.* $5\frac{1}{4}$*d.* $\times \frac{19}{27}$; $\frac{361}{880}$ of 1 acre.
6. 3 miles 5 furlongs $\times \frac{15}{16}$; 5 quarters 4 bushels $\div 3\frac{7}{19}$.
7. $\frac{128}{225}$ of 11 weeks 5 days; 15 cubic yards 2 feet $\div 65\frac{7}{9}$.
8. $\frac{2}{3}$ of $5\frac{5}{8}$ of $\frac{13}{15}$ of $\frac{4}{7}$ of 13*s.* $1\frac{1}{2}$*d.*; $\dfrac{4\frac{2}{9}}{8\frac{1}{7}}$ of 1 day.
9. $\frac{3}{4} \times \dfrac{3\frac{1}{5} \times \frac{4}{9}}{\frac{14}{15} + \frac{2}{3}} \times 5\frac{1}{2}$ of £3 15*s.*; $\dfrac{3\frac{1}{7} - 2\frac{1}{3}}{4\frac{1}{11} - 2\frac{1}{4}} \times 4\frac{1}{2}$ of 1 cwt.
10. £$\frac{1}{4} + \frac{7}{15}$*s.* $+ 1\frac{2}{5}$*d.*; $\frac{11}{12}$ of £1 5*s.* $- \frac{3}{8}$ of 2*s.* 6*d.* $+ 1\frac{7}{8}$ of 13*s.* 4*d.* $- \frac{11}{60}$ of £5.

II. To reduce one quantity to the fraction of another. *Express both in the same denomination, and divide the first number by the second.*

1*s.* 4*d.* to fraction of 10*s.* 6*d.*

$$1s.\ 4d. = 16d.;\ 10s.\ 6d. = 126d.\quad \frac{16}{126} = \frac{8}{63}$$

4 cwt. 2 qr. to fraction of 15 cwt. 3 qr.

$$4 \text{ cwt. } 2 \text{ qr.} = 18 \text{ qr.};\ 15 \text{ cwt. } 3 \text{ qr.} = 63 \text{ qr.}\quad \frac{18}{63} = \frac{2}{7}$$

It is not necessary to choose such a denomination as will allow the quantities to be expressed in whole numbers. Thus, the preceding example might, perhaps, have been more conveniently worked by choosing the denominations of shillings and hundredweights respectively:—

$$1s.\ 4d. = 1\tfrac{1}{3}s.;\ 10s.\ 6d. = 10\tfrac{1}{2}s.\quad 1\tfrac{1}{3} \div 0\tfrac{1}{2} = \frac{4}{3} \times \frac{2}{21} = \frac{8}{63}$$

$$4 \text{ cwt. } 2 \text{ qr.} = 4\tfrac{1}{2} \text{ cwt.};\ 15 \text{ cwt. } 3 \text{ qr.} = 15\tfrac{3}{4} \text{ cwt.}$$

$$4\tfrac{1}{2} \div 15\tfrac{3}{4} = \frac{9}{2} \times \frac{4}{63} = \frac{2}{7}$$

For the sake of explanation, much more of the work has here been shown than is practically necessary. In future examples this will be avoided. The only difficulty in applying this rule is that it requires some little judgment to choose the denomination that will admit of the question being worked out in the quickest and neatest manner possible. To illustrate this point an example is worked out in two different ways.

Reduce £1 8*s*. $1\frac{1}{2}$*d*. to the fraction of £3 8*s*. 9*d*.

First method.

28*s*. $1\frac{1}{2}$*d*.	68*s*. 9*d*.
12	12
337	825
2	2
675	1650

$$\frac{675}{1650}=\frac{27}{66}=\frac{9}{22}$$

Second method.

$$28\tfrac{1}{8}\div68\tfrac{3}{4}=\frac{225}{8}\times\frac{4}{275}=\frac{9}{22}$$

Additional Examples: 6 poles 2 yd. to the fraction of 1 mile 6 fur.

$$6\tfrac{4}{11}\div(14\times40)=\frac{70}{11}\times\frac{1}{14}\times\frac{1}{40}=\frac{1}{88}$$

What fraction is $2\frac{4}{5}$ acres of $\frac{7}{10}$ of a square mile? $\frac{14}{5}\times\frac{10}{7}\times\frac{1}{640}=\frac{1}{160}$

3 weeks 4 days 11 hours 15 minutes to the fraction of 3 hr. 20 m.

$$25\tfrac{15}{16}\div3\tfrac{1}{3}=\frac{415}{16}\times\frac{3}{10}=\frac{249}{32}=7\tfrac{25}{32}.$$

$\frac{2}{3}$ of $1\frac{7}{8}$ of 1000 pecks to the fraction of $3\frac{1}{8}$ quarters.

$$\frac{2}{3}\times\frac{15}{8}\times\frac{1000}{1}\times\frac{1}{4}\times\frac{1}{8}\times\frac{8}{25}=\frac{25}{2}=12\tfrac{1}{2}.$$

Ex. 23.

Reduce

1. 6*s*. $9\frac{1}{4}$*d*. to the fraction of £1; 7*s*. 6*d*. to the fraction of 10 guineas.
2. 3*s*. $8\frac{1}{4}$*d*. to the fraction of 1*s*. $6\frac{3}{4}$*d*.; £3 5*s*. 2*d*. to the fraction of £12 15*s*.
3. 1 cwt. 2 qr. 12 lb. to the fraction of 1 ton; 7 oz. 9 dwt. 8 gr. to the fraction of 1 lb. troy.
4. 7 miles 3 fur. 16 po. to the fraction of 1 league; 5 days 11 hr. 15 min. to the fraction of 1 week.
5. 2800 fourpenny pieces to the fraction of $6\frac{1}{4}$ guineas; 32 acres. 1 rd. 24 per. to the fraction of 1 square mile.
6. $1\frac{2}{5}$ of 1 lb. 10 dwts. troy to the fraction of 1 lb. avoirdupois; 300 scruples to the fraction of 1 lb. 10 dwts. troy.
7. $2\frac{5}{11}$ of $\frac{28}{81}$ of 10 qrs. 5 bus. to the fraction of $\frac{2}{3}$ of $8\frac{1}{3}$ of 70 pecks; 2 ft. $5\frac{1}{3}$ in. to the fraction of 1 chain.
8. 11000 sq. ft. to the fraction of $\frac{5}{8}$ of $2\frac{1}{2}$ acres; £11 15*s*. $4\frac{1}{4}$*d*. to the fraction of £44 18*s*. $7\frac{1}{2}$*d*.
9. $\frac{7}{12}$ of 3 fur. 9 po. 4 yd. 1 ft. 6 in. to the fraction of 7 po. 1 yd.; 3 roods 17 per. 7 yd. to the fraction of 5 acres 20 per.
10. $2\frac{7}{9}$ of 41 wks. 5 days to the fraction of $6\frac{2}{3}$ of 7 years; 700 nails to the fraction of 3 poles 1 yard.

III. Since decimals are capable of being reduced to fractions, and are also merely an extension of the ordinary notation used for whole numbers, it follows that when decimals are applied to concrete quantities, the questions arising therefrom may generally be treated in two different ways. Either the decimals may be reduced to fractions, in which case the examples may be worked by the rules already given, or a method may be employed based upon the similarity between decimals and whole numbers.

To find the value of a decimal of any quantity: *The quantity being reduced to one denomination, multiply it and the decimal together, reduce the decimal part of the product to the next lower denomination, and continue reducing the decimal parts of each denomination to the next lower, as far as may be necessary.*

Or, *Reduce the decimal to a fraction, and find the value of that fraction of the given quantity.*

Examples. 3·0775 of £7

$$\begin{array}{r} 3{\cdot}0775 \\ 7 \\ \hline 21{\cdot}5425 \\ 20 \\ \hline 10{\cdot}85\cancel{00} \\ 12 \\ \hline 10{\cdot}2\cancel{0} \end{array}$$

£21 10*s.* 10·2*d.*

28·396 of £1 7*s.* 6*d.*

£1 7*s.* 6*d.* = 27·5*s.*

$$\begin{array}{r} 28{\cdot}396 \\ 27{\cdot}5 \\ \hline 141980 \\ 198772 \\ 56792 \\ \hline 780{\cdot}89\cancel{00} \\ 12 \\ \hline 10{\cdot}68 \end{array}$$

£39 0*s.* 10.68*d.*

3·625 of £14 13*s.* 6*d.*

$3{\cdot}625 = 3\frac{5}{8}$

$$\begin{array}{rrr} 14 & 13 & 6 \\ & & 3 \\ \hline 44 & 0 & 6 \\ 5 & 10 & 0\frac{3}{4} \\ 3 & 13 & 4\frac{1}{2} \\ \hline 53 & 3 & 11\frac{1}{4} \end{array}$$

N. B. In this example £5 10*s.* $0\frac{3}{4}$*d.* = $\frac{3}{8}$ of £14 13*s.* 6*d.*; obtained by dividing the line above by 8 and £3 13*s.* $4\frac{1}{2}$*d.* = $\frac{1}{4}$ of £14 13*s.* 6*d.* $\frac{3}{8} + \frac{1}{4} = \frac{5}{8}$.

In the last of these examples it was better to reduce the decimal to a fraction.

Ex. 24.

Find the values of

1. 3·75 of 7*s.* 4*d.*; ·1024 of £6 10*s.* 2½*d.*
2. ·3125 of £1; 1·875 of 1 guinea.
3. ·046875 of 10*s.*; ·71875 of £2 5*s.* 4*d.*
4. ·1608 of 1 cwt. 1 qr. 16¼ lbs.; ·59375 of 1 bushel.
5. 2·6931 of 1 lb. troy; ·00176 of £10.
6. 3·1872 of 3 weeks 4 days; 1·869140625 of 1 ton.
7. 6·237 of 1 mile; 3·2784 of 20 perches 20 sq. yds.
8. 19·3865 of a cubic yard; $2·5 \times \frac{3}{16} \times 1·6$ of 9*s.* 4*d.*

IV. To reduce a quantity to the decimal of another quantity. One general class of examples of this kind is where a quantity expressed in several denominations has to be reduced to the decimal of another quantity expressed in a higher denomination. To do this: *Reduce the number of the lowest denomination to the decimal of the next higher, and prefix to that decimal the whole number, if any, of that denomination. Continue this process as far as necessary.*

Examples. £3 8*s.* 2¼*d.* to the decimal of £5.

```
12)2·25
20)8·1875
 5)3·409375
    ·681875
```

4 fur. 52 yd. 9 in. to the decimal of 6 miles 2 fur.

6 m. 2 fur. = 50 furlongs.

```
11)52·25
20) 4·75
50) 4·2375
    ·08475
```

Or, *Reduce the quantity to the fraction of the other, and reduce that fraction to a decimal.*

This is generally the best method when the second quantity is expressed in several denominations.

Reduce £1 13*s.* 11*d.* to the decimal of £11 14*s.* 8*d.*

$$33\tfrac{11}{12} \div 234\tfrac{2}{3} = \frac{407}{12} \times \frac{3}{704} = \frac{37}{256} = ·14453125$$

```
8)37
 8)4·625
  4)·578125
    ·14453125
```

Ex. 25.

Reduce

1. 15*s.* 9*d.* to dec. of £1; 7*s.* 6¾*d.* to dec. of 5*s.*
2. £1 6*s.* 3*d.* to dec. of 4 guineas; 300 farthings to dec. of £5.

3. 13 cwt. 2 qr. 13 lb. 9 oz. to dec. of 1 ton; 90 lb. av. to de
1 lb. troy.

4. 16 hr. 29 m. to dec. of 1 day 4 hr. 40 m.; £7 0*s.* 0¼*d.* to de
£4 6*s.* 2*d.*

5. ·0132 of 3 lb. 8 oz. to dec. of 1 cwt.; 3 days 5 hr. 42 m. to de
1 week.

6. £18 3*s.* 9¾*d.* to dec. of 2*s.* 6*d.*; 6 fur. 18 po. 2 yd. 2 ft. 3 i
dec. of 1 mile.

7. 60·25 of 5 roods to dec. of 10 acres; 3 qr. 2 bus. 1 gal. 3 qt.
to dec. of 1 load.

8. 3·125 of 1 cwt. to dec. of 1 qr. 12 lb.; 29160 cub. in. to de
cub. yard.

Where circulating decimals occur in connection with c
crete quantities, the general rule is to reduce them
fractions, as for example: Find the value of $\cdot 2\dot{8}5365\dot{5}$
£2 8*s.* 4*d.*:

$$\cdot 2\dot{8}5365\dot{5} = \frac{285363}{999990} = \frac{31707}{111110} = \frac{117}{410}$$

£2 8*s.* 4*d.* = 580*d.*

$$\frac{117}{410} \times \frac{580}{1} = \frac{6786}{41} = 165\tfrac{21}{41}d. = 13s.\ 9\cdot 5121\dot{9}d.$$

```
41)21·0(51219
   20 5
   ----
     50
     41
     --
     90
     82
     --
      80
      41
      --
      390
      369
      ---
       21
```

But it is possible frequently, even in cases where the per
is as long as in the example just given, to obtain the res
more easily by putting down a sufficient number of figu
of the original decimal to enable us to see what the cir
lating period is in the result. Thus, multiplying $2\dot{8}5365\dot{5}$
580—

```
·28536585365585
            580
---------------
 22829268292926
1426829268292
---------------
165512195121 8
```

and the product is evidently $165\cdot 5121\dot{9}$.

In the instances, however, which most frequently occur in practice the circulating decimals are of short period or are equivalent to fractions with a small denominator, and the fraction equivalent to the circulating period may be placed instead of the period at the end of the decimal.

Thus. $\cdot 0387\dot{6}$ of £2; $\cdot 256\dot{4}2857\dot{1}$ of a lb. troy; and $\cdot 6950\dot{7}\dot{5}$ of a mile.

$$\begin{array}{r}
\cdot 0387\frac{2}{3} \\
40 \\ \hline
1\cdot 550\frac{2}{3} \\
12 \\ \hline
6\cdot 608
\end{array}
\quad 1s.\ 6\cdot 608d.$$

$$\left.\begin{array}{r}
\cdot 256\frac{3}{7} \\
12 \\ \hline
3\cdot 077\frac{1}{7} \\
20 \\ \hline
1\cdot 54\frac{2}{7} \\
12 \\ \hline
6\cdot 51\frac{3}{7} \\
2 \\ \hline
13\cdot 02\frac{6}{7}
\end{array}\right\}
\begin{array}{llll}
 & \text{oz.} & \text{dwt.} & \text{gr.} \\
 & 3 & 1 & 13\cdot 02\dot{8}57142\dot{2} \\
 & \text{oz.} & \text{dwt.} & \text{gr.} \\
\text{or} & 3 & 1 & 13\cdot 0\dot{2}8571\dot{4}
\end{array}$$

$$\begin{array}{r}
\cdot 6950\frac{25}{33} \\
8 \\ \hline
5\cdot 5606\frac{2}{33} \\
11 \\ \hline
6\cdot 1666\frac{2}{3} \\
20 \\ \hline
123\cdot 333\frac{1}{3} \\
3 \\ \hline
1\cdot 000
\end{array}
\qquad
\begin{array}{ccc}
\text{fur.} & \text{yds.} & \text{ft.} \\
5 & 123 & 1
\end{array}$$

In examples of the reduction of concrete quantities to the decimals of other quantities, it often happens that circulating decimals occur in consequence of dividing by some factor other than 2 or 5, which does not occur in the number divided. In these cases it is generally sufficient to put down a few places of the decimal, and the resulting circulator will be easily discovered.

Reduce 3 hr. 34 m. 32 sec. to the decimal of 1 day.
100 feet to the decimal of 1 mile.
£1 2s. 8d. to the decimal of £7.

$$\begin{array}{l}
60)32\cdot 000.... \\
60)34\cdot 5333.... \\
\left\{\begin{array}{l} 8)\ 3\cdot 57555.... \\ 3)\ \cdot 44694444.... \end{array}\right. \\
\quad \cdot 1489814814.... \\
\quad \cdot 14898\dot{1}\dot{4}
\end{array}
\qquad
\begin{array}{l}
3)100\cdot 000.... \\
\left\{\begin{array}{l} 20)\ 33\cdot 333.... \\ 11)\ 1\cdot 6666.... \end{array}\right. \\
8)\ \cdot 151515.... \\
\quad \cdot 0189393.... \\
\quad \cdot 018\dot{9}\dot{3}
\end{array}
\qquad
\begin{array}{l}
12)8\cdot 0000.... \\
20)2\cdot 6666.... \\
7)1\cdot 13333333.... \\
\quad \cdot 161904761.... \\
\quad \cdot 1\dot{6}1904\dot{7}
\end{array}$$

It may be noticed that in Avoirdupois weight there is only the factor 7 occurring in 28 lbs. = 1 qr. that can produce a circulating decimal, and the decimal will have a

period of 6 figures. In the Table of Measure of Length between inches and miles there is the factor 3 in 12 inches= 1 foot; 3 again in 3 feet=1 yard; 11 in 220 yards=1 furlong. In Square Measure 9 occurs between inches and feet, and also between feet and yards, and 121, giving 22 figures, between yards and roods. In the Table of Money there is the factor 3 in 12 pence=1 shilling. In the Table of Time the factor 3 occurs in 60 seconds=1 minute; in 60 minutes =1 hour, and in 24 hours=1 day. The factor 7 occurs in 7 days=1 week; and the factor 73 in 365 days=1 year. This last factor produces, as we have already seen, a period of eight figures. From these considerations we can generally tell how many figures there will be in the circulating period of a decimal which would result from any given question. Taking as an instance a somewhat complicated example:— What would be the period of the decimal that 17 seconds is of 12 weeks? Here there is a factor 3 in passing from seconds to minutes, 3 from minutes to hours, 3 from hours to days, 7 from days to weeks, and 3 is contained in the given number 12. The resulting decimal must therefore be equal to a fraction whose denominator, besides the factors 2 and 5 repeated any number of times, contains also $3^4 \times 7$. Now 7 is a factor of 999999, and 3^4, or 81, is a factor of 999999999, and hence $3^4 \times 7$ will be a factor of the L. C. M. of these, or of the number formed by 9 repeated 18 times. There will therefore be 18 figures in the circulator. The decimal actually is $\cdot 00002342\dot{3}7213403880070546\dot{7}$.

The following are other instances of the same kind:—

3 yds. 2 ft. to decimal of a mile ... One figure.

16 dwts. 7 gr. to decimal of 150 lbs. Troy ... Three figures.

14 lbs. 8 oz. to decimal of 1 cwt. ... Six figures.

1000 seconds to decimal of 1 year ... Twenty-four figures.

The number of non-circulating decimals may also, if required, be found in the manner explained on page 82.

Ex. 26.

Find the values of

1. $\cdot 491\dot{6}$ of £1 ; $\cdot 661458\dot{3}$ of £5.
2. $\cdot 303\dot{5}7142\dot{8}$ of 3 guineas ; $\cdot 569\dot{4}$ of 5*s.*
3. $\cdot 405\dot{3}\dot{0}$ of 1 mile ; $5\cdot\dot{7}0\dot{3}$ of 1 day.
4. $\cdot 507\dot{7}1604938\dot{2}$ of 1 square yard ; $\cdot\dot{2}191780\dot{8}$ of $\cdot 0125$ of 1 year.
5. $\cdot\dot{6}4\dot{8} \times \cdot 1\dot{6} \times \cdot 875 \times \dot{3}$ of 6*s.* 2*d.* ; $\cdot 897\dot{7}\dot{2}$ of £2 10*s.*

Reduce

6. 8*s.* 6½*d.* to dec. of £1 ; 4 oz. 16 dwt. $7\cdot\dot{1}$ gr. to dec. of 1 lb. Troy.
7. 2 qr. 3 lb. 12 oz. to dec. of 1 cwt. ; £41 8*s.* 9*d.* to dec. of £17 8*s.* 6*d.*
8. 3 qt. $0\cdot\dot{9}1\dot{8}$ pt. to dec. of 1 peck ; £4 6*s.* 4*d.* to dec. of £10 11*s.* 2½*d.*
9. 26 cub. ft. 576 in. to dec. of cub. yd. ; 3 acres 1 rood 16 per. $6\cdot 26\dot{5}1$ yds. to dec. of 7 acres 17 per. 2 yd.
10. 5 oz. 9 dwt. 4 gr. to dec. of 1 lb. Av. ; 19 min. 28 sec. to dec. of year.

Find, by reference to the particulars on page 82 and without calculating the decimals themselves, how many non-circulating and how many circulating figures there are in the decimals equivalent to

11. $\dfrac{\text{15 pounds}}{\text{11 tons}}$ 12. $\dfrac{\text{100 days}}{\text{7 years}}$ 13. $\dfrac{3s.\ 6d.}{\text{20 guineas}}$

14. $\dfrac{\text{80 feet}}{\text{1 league}}$ 15. $\dfrac{\text{500000 sq. in.}}{\text{1 acre}}$ 16. $\dfrac{\text{2000 lbs.}}{\text{2 tons 1 cwt.}}$

CHAPTER V.

PRACTICE AND PROPORTION.

In Chapter II. it was stated that examples in compound multiplication could, in general, be more easily worked out by a method called 'Practice.' This process, which could not be properly explained until the student had acquired a knowledge of fractions, is termed 'Practice' because of its very great use in the affairs of ordinary life. To show what it is let that example in Chapter II. be considered which comes immediately before the statement referred to. It is to find the value of 7 dwts. 2 gr. × 1427, and is worked out in two ways, firstly, by multiplying 7 dwts. 2 grs., not reduced to one denomination, by 1427; secondly, by reducing 7 dwts. 2 grs. to 170 grains, multiplying 1427 and 127 grains together, and reducing the product to pounds and ounces. A third way would be to multiply 1427 by the unreduced 7 dwts. 2 grs., and this is the process now to be explained.

To effect this multiplication we consider the multiplicand as a concrete number of the highest denomination, multiply it by the number of that denomination in the multiplier, and add such fractions of it as the numbers of lower denominations are of one of the highest. In the case of 1427 × 7 dwts. 2 grs., consider 1427 as pennyweights, multiply by 7, add $\frac{1}{12}$ of the multiplicand, because 2 grs. $=\frac{1}{12}$ dwt., and reduce to pounds and ounces. The work would be expressed thus:—

	1427 × 7 dwt. 2 gr.		
	7		
	9989		
2 gr. = $\frac{1}{12}$ dwt.	118	22	42 lb. 1 oz. 7 dwt. 22 gr.
	20) 10107	22	
	12) 505	7	
	42	1	

The above example is given for the purpose of showing ıat Practice is applicable to concrete quantities of any ind, weight, measure, length, &c., and is not confined to uestions involving money. At the same time such quesons are those which most frequently occur — so much so, ıdeed, that in many works on Arithmetic, Practice is dened to be the mode of finding the *value* of any number of ıings. The following example was also worked out in 'hapter II. by Compound Multiplication :—

	527 × £1 15s. 3½d.		
10s. = ½ of £1	263	10	
5 = ½ of 10s.	131	15	
3d. = $\frac{1}{20}$ of 5s.	6	11	9
½ = ⅙ of 3d.	1	1	11½
	929	18	8½

Before proceeding to work out any more examples, two r three particulars to be noted are given.

I. It is especially important in writing down an example, ıat the particular concrete multiplier corresponding to ach line to be added should be in the same straight line rith it. Attention to this point tends to prevent mistakes, nd keeps the work in a form which renders revision easy, rhile neglect of it produces confusion.

II. The denomination of which each multiplier is a raction may be left out, and understood instead of being xpressed. Thus, instead of writing in the third line of ıe above example *5s.* = ½ of 10*s.*, 131 15, it would be ufficient to write 5 ½ 131 15; and the multipliers and

the fractions are often kept in two ruled columns at the side. The example would then be written :—

		527 × £1 15*s*. 3½*d*.		
10*s*.	$\frac{1}{2}$	263	10	
5	$\frac{1}{2}$	131	15	
3*d*.	$\frac{1}{20}$	6	11	9
$\frac{1}{2}$	$\frac{1}{6}$	1	1	11½
		929	18	8½

III. Sometimes the multiplicand may be considered as a concrete number not of the highest denomination of the multiplier, but of a denomination higher than that. Thus in the question, What is the value of 549 things at 14*s*. 7*d*.? instead of considering 549 as shillings, and multiplying by 14, it would be better to consider 549 as pounds, take $\frac{1}{2}$ for 10*s*., $\frac{1}{5}$ for 4*s*., and so on.

Examples worked out.

		51427 at 15*s*. 7¾*d*.		
10*s*.	$\frac{1}{2}$	25713	10	
5	$\frac{1}{2}$	12856	15	
6*d*.	$\frac{1}{10}$	1285	13	6
1½	$\frac{1}{4}$	321	8	4½
¼	$\frac{1}{6}$	53	11	4¾
		£40230 18*s*. 3¼*d*.		

		1853 at £3 14*s*. 9¼*d*.		
		3		
		5559		
10*s*.	$\frac{1}{2}$	926	10	
4	$\frac{1}{5}$	370	12	
8*d*.	$\frac{1}{6}$	61	15	4
1	$\frac{1}{8}$	7	14	5
¼	$\frac{1}{4}$	1	18	7¼
		£6927 10*s*. 4¼*d*.		

		cwt.	qr.	lb.	oz.
		147 × 2	3	12	8
		2			
		294			
2 qr.	$\frac{1}{2}$	73	2		
1	$\frac{1}{2}$	36	3		
8 lb.	$\frac{1}{7}$	10	2		
4	$\frac{1}{2}$	5	1		
8 oz.	$\frac{1}{8}$		2	17 lb.	8 oz.
		420	2	17	8

		237 at £2 19*s*. 6½*d*.			
4*d*.	$\frac{1}{60}$	3	19		
1	$\frac{1}{4}$		19	9	
½	$\frac{1}{2}$		9	10½	
		5	8	7½	= price at 5½*d*.
		711			= price at £3
		705	11	4½	

Ex. 27.

Find the values of

1. £1 6*s*. 10½*d*. × 839 ; £5 2*s*. 8*d*. × 977.
2. £4 15*s*. 8½*d*. × 581 ; £7 18*s*. 1¼*d*. × 275.
3. 11*s*. 4½*d*. × 158 ; £2 9*s*. 7*d*. × 1049.
4. £6 13*s*. 1¾*d*. × 2087 ; 10*s*. 8½*d*. × 5042.

5. £17 8*s.* 9¾*d.* × 263 ; £4 2*s.* 8¼*d.* × 6076.
6. 7*s.* 7½*d.* × 1093 ; £3 17*s.* 2¾*d.* × 497.
7. £7 19*s.* 10½*d.* × 8653 ; 4*s.* 5½*d.* × 249.
8. 4 cwt. 2 qr. 8 lb. × 793 ; 2 lb. Troy 11 oz. 5 dwt. × 865.
9. 2 gal. 2 qt. 1 pt. × 947 ; 17 yd. 2 ft. 7 in. × 1028.
10. 21 days 9 hr. 35 min. 27 sec. × 563 ; 5 m. 2 fur. 11 po. × 719.
11. 1 qr. 5 lb. 7 oz. 11 dr. × 850 ; 3 sq. yd. 5 ft. 97 in. × 1096.

A question about price may assume a more complicated form, when the quantity is expressed as a concrete number as well as the value. For example, let it be required to find the value of 139 cwt. 3 qr. 10 lbs. 7 ounces, at £3 14*s.* 8*d.* per cwt. Here a double application of Practice is required, first to find the value of 139 cwt. at £3 14*s.* 8*d.*, and next to take such fractions of £3 14*s.* 8*d.* as will give the value of 3 qr. 10 lbs. 7 oz. This and three other examples are here worked out.

139 cwt. 3 qr. 10 lb. 7 oz. at £3 14*s.* 8*d.* per cwt.

		3		
		417		
10*s.*	$\frac{1}{2}$	69	10	
4	$\frac{1}{5}$	27	16	
8*d.*	$\frac{1}{6}$	4	12	8
2 qr.	$\frac{1}{2}$	1	17	4
1	$\frac{1}{2}$		18	8
8 lb.	$\frac{1}{7}$		5	4
2	$\frac{1}{4}$		1	4
4 oz.	$\frac{1}{8}$			2
2	$\frac{1}{2}$			1
1	$\frac{1}{2}$			$\frac{1}{2}$
		522	1	7½

1084 qr. 5 bus. 3 pk. at £2 2*s.* 8*d.* per qr.

		2		
		2168		
2*s.*	$\frac{1}{10}$	108	8	
8*d.*	$\frac{1}{3}$	36	2	8
4 bus.	$\frac{1}{2}$	1	1	4
1	$\frac{1}{4}$		5	4
2 pk.	$\frac{1}{2}$		2	8
1	$\frac{1}{2}$		1	4
		2314	1	4

19 lb. Troy 7 oz. 13 dwt. 8 gr. at
per l

10s.	$\frac{1}{2}$	9	10	
5	$\frac{1}{2}$	4	15	
6d.	$\frac{1}{10}$		9	6
3d.	$\frac{1}{2}$		4	9
6 oz.	$\frac{1}{2}$		7	$10\frac{1}{2}$
1 oz. 10 dwt.	$\frac{1}{4}$		1	$11\frac{5}{8}$
3 dwt.	$\frac{1}{10}$			$2\frac{29}{80}$
8 gr.	$\frac{1}{9}$			$\frac{21}{80}$
		15	9	$3\frac{3}{4}$

17 miles 3 fur. 136 yd. at £4
4 per mil

		68		
2s.	$\frac{1}{10}$	1	14	
8d.	$\frac{1}{3}$	0	11	4
2 fur.	$\frac{1}{4}$	1	0	8
1	$\frac{1}{2}$		10	4
110 yd.	$\frac{1}{2}$		5	2
22 yd.	$\frac{1}{5}$		1	$0\frac{2}{5}$
2	$\frac{1}{11}$			$1\frac{7}{55}$
2	1			$1\frac{7}{55}$
		72	2	$8\frac{36}{55}$

Ex. 28.

Find the values of

1. 23 tons 2 cwt. 1 qr. 8 lb. at £5 7*s*. 4*d*. per ton.
2. 135 cwt. 2 qr. 13 lb. at £3 1*s*. 10*d*. per cwt.
3. 173 acres 3 roods 29 poles at £82 13*s*. 4*d*. per acre.
4. 57 lb. Troy 11 oz. 17 dwt. 12 gr. at £3 15*s*. 4*d*. per lb.
5. 73 qr. 5 bus. 7 gal. 2 qt. at 13*s*. 4*d*. per bushel.
6. 83 miles 5 fur. 35 yd. at £12 7*s*. 6*d*. per league.
7. 17 square yards 8 ft. 37 in. at 3*s*. $4\frac{1}{2}$*d*. per yard.
8. 3 weeks 4 d. 11 hr. 25 m. at £5 5*s*. per week.
9. 5 cwt. 3 qr. 11 lb. 10 oz. at £1 3*s*. 4*d*. per quarter.
10. 14 miles 5 fur. 97 yd. 2 ft. at £1 7*s*. 6*d*. per furlong.
11. 6 miles 3 fur. 19 po. 3 yd. at £71 10*s*. per mile.
12. 15 acres 3 roods 66 yd. at £3 6*s*. per square chain.
13. What is the amount of a bankrupt's dividend on £1428 if he pays 8*s*. 8*d*. in the £?
14. What is the poor-rate on a rental of £1157 10*s*. at 2*s* the £?

With 'Practice' finishes the consideration of wh

be called the framework of arithmetic, the employment of the four processes of addition, subtraction, multiplication, and division. We have seen these applied to abstract numbers, to fractions, to decimals, to concrete numbers, including the particular instances of square and cubic measure, and have found that the multiplication of concrete numbers may be performed in three different ways: firstly, by reducing to one denomination, and using multiplication of abstract numbers; secondly, by making the unreduced concrete quantity the multiplicand, and using compound multiplication; thirdly, by substituting for compound multiplication the shorter method of practice.

The remaining part of arithmetic has for its subject the various combinations of these four processes, which are found practically useful. Of these combinations the most important is that called 'Proportion,' or the 'Rule of Three.' This we now proceed to explain.

The result of dividing 3 by 4, which, as we have seen, may be expressed as $3 \div 4$, $\frac{3}{4}$, or $3 : 4$, is sometimes called the *ratio* of 3 to 4. It is evident that ratios may be equal to one another. Thus $\frac{3}{4}$, $\frac{9}{12}$, $\frac{30}{40}$, are all equal ratios. The equality of two ratios is termed a *Proportion*, which consequently will consist of four numbers, two in each ratio. Thus $\frac{3}{4}=\frac{9}{12}$, or $3 : 4 = 9 : 12$, is a proportion. When the sign of division $\div$ is used, it is customary to substitute $::$ for $=$, and thus the proportion would be expressed $3 : 4 :: 9 : 12$, which is read, 'Three is to four as nine is to twelve,' and means the same as $\frac{3}{4}=\frac{9}{12}$. The numbers 3, 4, 9, 12 are called the *terms* of the proportion. From the equation $\frac{3}{4}=\frac{9}{12}$ we have, by multiplying by 4, $3 = \frac{4 \times 9}{12}$, and by again multiplying by 12, $3 \times 12 = 4 \times 9$. This relation between the terms is expressed thus: *The product of the extremes is equal to the product of the means*; in other words, 1st term $\times$ 4th term $=$ 2nd term $\times$ 3rd term.

Again, since $3 \times 12 = 4 \times 9$, therefore $12 = \frac{4 \times 9}{3}$ or 4th term$=\frac{\text{2nd term} \times \text{3rd term}}{\text{1st term}}$, and these equations may also be represented in the slightly different form $12=9 \times \frac{4}{3}$, or 4th term$=$3rd term$\times \frac{\text{2nd term}}{\text{1st term}}$. Hence it is clear, that if we know three terms of a proportion, the fourth may be found. For example, to find the fourth terms of the proportions of which the first three terms are respectively 12, 7, 60; 6, 5, 21; 75, 40, 15; 13, 19, 28. Applying the general rule, 4th term$=$3rd term$\times \frac{\text{2nd term}}{\text{1st term}}$, we have, in the first case, 4th term$=60 \times \frac{7}{12}=35$; in the second, 4th term$=21 \times \frac{5}{6}=\frac{35}{2}=17\frac{1}{2}$; in the third, 4th term$=15 \times \frac{40}{75}=8$; in the fourth, 4th term$=28 \times \frac{19}{13}=\frac{532}{13}=40\frac{12}{13}$.

Questions in Rule of Three are of this kind, where the existence of a proportion between the terms is taken for granted, and the first, second, and third terms are given. That there is such a proportion is generally known from some familiar rule in life, or some fact in nature. Thus, if we buy bread or meat, we know that the price paid is dependent upon the quantity bought; that the more we buy the more we have to pay, and the less we buy the less we have to pay; that if the quantity be doubled the price is doubled, and that in general the ratio of any two prices is equal to the ratio of the corresponding quantities. For example, if beef be 9*d.* per lb., 3 lb. will cost 2*s.* 3*d.* and 5 lb. will cost 3*s.* 9*d.*, and $\frac{3\text{ lb.}}{5\text{ lb.}}=\frac{2s.\ 3d.}{3s.\ 9d.}$ Again, from an investigation into the properties of fluids, we learn that the pressure at any point of a fluid is proportional to the depth below the surface, and hence, if we know that the pressure at 256 feet below the surface of the sea is 133 lbs. on the square inch, we conclude that at the depth of 60 feet the

pressure must be $133 \times \frac{60}{256}$, or $31\frac{11}{64}$ lbs. In the complete proportion there would, of course, exist the relation $\frac{256 \text{ feet}}{60 \text{ feet}} = \frac{133 \text{ lbs.}}{31\frac{11}{64} \text{ lbs.}}$ On the other hand, the question, 'If a body falls 64 feet in 2 seconds, how far will it fall in 8 seconds?' is not to be determined in the same way as the previous questions, because the distance a body falls is not proportional to the time of falling, and consequently $64 \times \frac{8}{2}$, or 256, would not be a correct answer.

When, however, we are satisfied that a proportion does exist, and three terms are given, the rule that the 4th term $= 3\text{rd term} \times \frac{2\text{nd term}}{1\text{st term}}$ at once enables us to find the 4th term.

The only difficulty attending the practical application of this rule, is that of determining which terms we are to consider as the 1st, 2nd, 3rd respectively. An example will illustrate this point clearly. Let the question proposed be, If 18 lb. of cheese cost 13 shillings, what will be the value of $1\frac{1}{2}$ cwt.? Now let it be observed, firstly, that there are two terms in what may be described as the *supposition* contained in the question, namely, 18 lbs. and 13 shillings; and, secondly, that of these two terms, one, namely, the 13 shillings, is of the same kind, money, as we should expect the answer to be, while the other is of the same kind as the single term $1\frac{1}{2}$ cwt. contained in what may be called the *demand*. The general rule applicable to this and all cases is, *The two terms in the supposition are the* 1*st and* 3*rd, the* 3*rd being that one which is of the same kind as the answer.* Hence, in the question proposed, 13 shillings is the 3rd term, 18 lbs. the 1st, and consequently $3\frac{1}{2}$ cwt. must be the 2nd. Therefore the answer, or 4th term, $= 13 \text{ shillings} \times \frac{1\frac{1}{2} \text{ cwt.}}{18 \text{ lbs.}} = \frac{13s.}{1} \times \frac{3}{2} \times \frac{112}{18} = \frac{364s.}{3} = 121\frac{1}{3}s. = £6\ 1s.\ 4d.$ Again,

If a family eat $43\frac{1}{2}$ loaves in 29 days, how many loaves will they eat in 56 days? Here $43\frac{1}{2}$ loaves, being of the same kind as the answer, is the 3rd term, 29 days, the other term in the supposition, is the 1st term, and 56 days is the 2nd term.

Hence the answer, or 4th term, $=43\frac{1}{2}$ loaves $\times \frac{56 \text{ days}}{29 \text{ days}} = \frac{87}{2} \times \frac{56}{29}$ loaves $=84$ loaves.

It will be seen, then, that in all cases of ordinary or *direct* proportion the following rule holds good:—

$$\text{Answer or 4th term} = \text{term like answer} \times \frac{\text{term in demand}}{\text{term in supposition}}.$$

Ex. 29.

1. If cheese is $9\frac{1}{2}d.$ per lb., what is the value of 17 cwt. 3 qr. 10 lb.?
2. If household expenses are £122 6*s.* 7*d.* for 22 weeks 3 days, what will they be for 3 weeks 5 days?
3. If a coach-wheel make 1125 revolutions in 10000 feet, how many is that per mile?
4. A train is going at the rate of 40 miles an hour; through how many feet does it move in a second?
5. If the tax upon property assessed at £118 10*s.* is £10 12*s.* $3\frac{3}{4}d.$, what must be the assessment when the tax is £2 16*s.* $5\frac{1}{4}d.$?
6. The shadow of a perpendicular tower is observed to be 9 feet 4 inches long, at the same time that a stick 6 feet in length casts a shadow of $10\frac{1}{2}$ inches. Find the height of the tower.
7. Fifteen degrees of west longitude make a difference of one hour earlier in time. What will be the time at New York, the longitude of which is 73° 55′, when it is noon at Greenwich?
8. A man travelling uniformly on a journey of 10 miles passes the 7th milestone at 3.30, and comes to his journey's end at 21 min. past 4. At what time did he start?
9. A servant's wages being £22 a year, what should be paid for 90 days' service?
10. If 6 chests of tea, each containing 84 lbs., cost £73 10*s.*, how much is that per cwt.?
11. What does a bankrupt pay in the £ if his creditors receive £376 5*s.* 6*d.* out of £2076?

It has been explained that in all the preceding examples, the fact of there being a proportion between the terms must be taken for granted, and the process called 'Rule of

Three' may then be applied. The Rule of Three may, however, be also applied to one other class of questions, where a different relation exists between the quantities given. What this relation is, may best be explained by taking an ordinary instance. Suppose that in a prison each man is allowed $1\frac{1}{4}$ lb. of bread per day. Then, when there are 100 prisoners, 1000 lbs. of bread would last eight days, and if there were only 80, it would last ten days. If the question, therefore, had been proposed, A certain quantity of bread is sufficient for 100 prisoners for 8 days; for how many days will it be sufficient for 80? it is evident that it could not be worked out by ordinary proportion. For if that were tried, we should have

$$4\text{th term} = \text{term like answer} \times \frac{\text{term in demand}}{\text{term in supposition}} =$$

$$8 \text{ days} \times \frac{80 \text{ men}}{100 \text{ men}} = 6\tfrac{2}{5} \text{ days.}$$

And this result would be contrary to common sense, because it would be equivalent to saying that the fewer prisoners there were, the more quickly would the provisions be consumed. And, moreover, common sense would tell us that the state of the case was the exact contrary of this, that the fewer prisoners there were, the less quickly would the provisions be consumed, so that the less the number of prisoners the greater the number of days. Also, it would tell us that this increase or decrease was regular; that twice the number of prisoners would finish the food in half the number of days; that if the prisoners were reduced to one-third of their number, the days would be multiplied by three; and so on. A relation of this kind is called *inverse* proportion, and the number of prisoners would be said to be *inversely* proportional to the number of days. Returning then to the question last proposed, we see that the answer must be more than 8 days, as the number of prisoners is diminished from 100 to 80, and it must be, moreover,

increased in just the proportion of 100 to 80. Consequently

$$4\text{th term} = 8 \text{ days} \times \frac{100 \text{ men}}{80 \text{ men}} = 10 \text{ days}.$$

This example leads at once to the general rule, that in *Inverse Proportion* 4th term = term like answer $\times \dfrac{\text{term in supposition}}{\text{term in demand}}$.

If one set of quantities are said to be inversely proportional to another, the true mathematical meaning is that the first are directly proportional to unity divided by the second. Thus, in the preceding example, prisoners being inversely proportional to days, we should have the direct proportion $= \frac{1}{100} : \frac{1}{80} :: 8$ days : 10 days. And thus, in direct proportion, we always have—

Term in supposition : Term in demand :: Term like answer : answer ; and in Inverse Proportion, we have—

$\dfrac{1}{\text{Term in supposition}} : \dfrac{1}{\text{Term in demand}} ::$ Term like answer : Answer. By multiplying extremes and means, and dividing by the first term in the latter of these proportions, we obtain the rule already given for Inverse Proportion; namely—

$$4\text{th term} = \text{term like answer} \times \frac{\text{term in supposition}}{\text{term in demand}}.$$

If 16 men reap a field in 36 working hours, how many men would have been required to reap it in 48 hours?

$$\text{Here 4th term} = 16 \text{ men} \times \frac{36 \text{ hours}}{48 \text{ hours}} = 12 \text{ men}.$$

If a railway train, going at the rate of 40 miles an hour, takes $7\frac{1}{2}$ hours from London to Carlisle, how long would a luggage train going at $18\frac{3}{4}$ miles an hour take to perform the same journey?

$$\text{Here 4th term} = 7\tfrac{1}{2} \text{ hours} \times \frac{40 \text{ miles}}{18\frac{3}{4} \text{ miles}} = \frac{15 \text{ hrs.}}{2} \times \frac{40}{1} \times \frac{4}{75} = 16 \text{ hrs.}$$

Ex. 30.

1. If a room require 32 yards of carpet, 27 inches wide, how many would be necessary of carpet 24 inches wide?

2. The wheels of a carriage are 3 ft. 4 in. and 2 ft. 1 in. in diameter. How many revolutions will the hind wheel make while the fore wheel makes 1600?

3. The staircase inside a church tower is at first intended to be composed of 240 steps each 9 inches high, but afterwards it is determined to reduce the height of the steps to $7\frac{1}{2}$ inches. How many will then b requisite?

4. A certain piece of work occupies a party of workmen $8\frac{3}{4}$ hours per day for 16 days. How many days would have been required if they had worked $9\frac{1}{3}$ hours per day?

5. Beaten gold being 19·2 times as heavy as water, find what fraction of an inch gold leaf is in thickness, when a grain is made to cover 56·7 square inches.

6. If the cost of carrying a ton of goods 185 miles be 12*s.* 6*d.*, how many cwt. could be taken 240 miles for the same money?

7. A man is 12 hours on a journey, walking at the rate of $3\frac{1}{2}$ miles an hour. How long would he have taken if he had walked 4 miles an hour?

8. A friend lends me £130 for 9 weeks. For how long ought I to lend him £504 to return the obligation?

For the sake of distinctness, the examples of Direct Proportion have hitherto been kept separate from those of Inverse Proportion. As, however, for practical purposes it is not merely requisite to be able to apply each rule, but also to determine which is applicable to any particular instance, several examples are subjoined, of both rules, mixed up together indiscriminately. It will be necessary, in working these, to settle, in the first place, whether the one kind of things is proportional or inversely proportional to the other. Thus work done is proportional to the number of men, to the time occupied, and to the amount done in a given time, as an hour or a day; because if you increase any of these three things, you increase the work done. But the number of men required to finish a given piece of work is inversely proportional to the time allowed, and to the amount done in an hour or a day; for if you increase either of these, you diminish the number of men necessary.

When it has been ascertained whether the question is

one of direct or inverse proportion, the term, like the answer, must be multiplied by a fraction of which the other terms are numerator and denominator, the term in the demand being numerator in direct, and the term in the supposition being numerator in inverse proportion.

Though the above is the best way of working these examples, still there is another plan which may perhaps be found simpler, and is applicable to both kinds of proportion. It is: *Increase or diminish the term like the answer in the ratio indicated by the other pair of terms.* Of course this rule implies that common sense must be used to settle whether the third term should be increased or diminished. The following are examples of this mode of considering questions in Proportion:—

If 19 cwt. 2 qr. of lead cost £21 12*s.* 3*d.*, how much is that per lb.?

Here the term, like the answer, is £21 12*s.* 3*d.*; and as the answer is evidently *less* than that, the term must be *diminished.*

$$\text{Hence answer} = £21\ 12s.\ 3d. \times \frac{1\text{ lb.}}{19\text{ cwt. } 2\text{ qr.}} = £21\tfrac{49}{80} \times \frac{1}{78} \times \frac{1}{28}$$

$$= \frac{1729}{80} \times \frac{240d.}{1} \times \frac{1}{78} \times \frac{1}{28} = 2\tfrac{3}{8}d.$$

If a man is paid half as much more than a boy, and 16 men and 5 boys earn £16 1*s.* 5*d.*, what would 100 boys earn in the same time?

The 16 men and 5 boys must earn as much as 29 boys, and the term like the answer, £16 1*s.* 5*d.*, must evidently be *increased.*

$$\text{Hence answer} = £16\ 1s.\ 5d. \times \frac{100}{29} = £16\tfrac{17}{240} \times \frac{100}{29} = \frac{£3857}{240} \times \frac{100}{29}$$

$$= \frac{£665}{12} = £55\ 8s.\ 4d.$$

It must be repeated, however, that this method of working is not so good, in many respects, as that previously given, namely, first determining the nature of the proportion, and then putting one or the other term in the numerator of the fraction accordingly.

Ex. 31.

1. If $3\frac{1}{2}$ oz. of gold dust are worth £12 5*s.*, what is the value of 1 lb. 5 oz. 7 dwt.?

2. If 17 cwt. of sugar cost £3 4*s.* 5*d.*, how much will 56 cwt. cost?

3. Sound travels at the rate of 1140 feet in a second. How long will it take to travel 2 miles 3 furlongs?

4. If 18 lb. of sugar cost 5*s.* $0\frac{3}{4}$*d.*, what will be the price of 2 cwt. 1 qr. 4 lb.?

5. Mercury is 13·58 times heavier than water. What is the height of a column of water that can be just supported by the pressure of the air, supposing the barometer to be at 30 inches?

6. A piece of land is divided into 113 allotments, each containing 1 rood 8 yards. How many would there have been if each had contained 1 rood 12 perches 9 yards?

7. If $\frac{7}{8}$ of $\frac{3}{4}$ of 5 cwt. cost £8 1*s.* $10\frac{1}{2}$*d.*, what is the value of $\frac{9}{16}$ of a ton?

8. If a gentleman pays income-tax at 7*d.* in the £, amounting to £17 9*s.* 5*d.*, what is his net income?

9. The express to the north reaches Carlisle, 300 miles from London, in $7\frac{3}{4}$ hours; how long would a luggage train going at $\frac{3}{8}$ of the speed take to perform the journey?

10. If $\frac{5}{9}$ of ·025 of $4\frac{1}{2}$ loads cost 2*s.* 6*d.*, how much is that per bushel?

11. If 1 lb. 6 oz. 3 dwt. 8 gr. of gold is worth £68 2*s.* 6*d.*, how much is that per oz.?

12. A French 'metre' being equal to 39·371 inches, find the number of metres in an English mile correct to three places of decimals.

13. If 6500 tiles $4\frac{1}{2}$ inches long and $2\frac{3}{4}$ inches broad be required to pave a certain space, what number would have been wanted of tiles $6\frac{1}{2}$ in. long and $5\frac{1}{2}$ in. broad?

14. A man buys a number of old books for £5, and expects to make £2 10*s.* profit by selling them at an average price of 8*d.* a volume. He actually only obtains $6\frac{1}{2}$*d.* a volume. What profit does he make?

15. A man who owns $\frac{3}{8}$ of a ship sells $\frac{3}{4}$ of his share for £250 to another who already owns $\frac{2}{5}$ of the ship. What is the value of the shares belonging to other persons?

16. A certain sum of money lasts 195 days when it is spent at the rate of £4 4*s.* a week. How long would it have lasted if spent at the rate of £2 12*s.* 6*d.* per week?

17. If 3·125 of 18·4 of 12 men can finish a piece of work in $5\frac{1}{4}$ days, in how many days would 23 men finish it?

18. If £16 4*s.* is paid for lodgings from July 10 to Nov. 22, what should be paid from Sep. 3 to Oct. 28?

19. 1365 feet of timber are sold for £20 9*s.* 6*d.*, one-fifth of which is profit. Find the cost price of 100 feet.

20. **If ·428571 of 3·851 of 13 cwt. 2 qr. of bread cost £16 18s., what should be the price of a 4 lb. loaf?**

Hitherto questions have been proposed in which the answer differs from the third term only in consequence of a change in some one circumstance. Thus, in the question, If 8 men dig a ditch 100 yards long in a day, how many men will be required to dig a ditch 350 yards long in the same time? the answer, which is 28 men, depends upon the 8 men given in the question, and upon a certain supposed change in the length of the ditch. Changes of other circumstances would, however, also alter the number of men requisite; as, for instance, a difference in the time allowed, the number of working hours in the day, the width and depth of the ditch. And several of such supposed changes might be contained in the same question, which would then become an example in *Compound Proportion*, or, as it is sometimes called, *Double Rule of Three.* As an example, take the question: If 8 men dig a ditch 100 yards long, 2 feet wide, and $1\frac{1}{4}$ deep, in 1 day of 10 working hours, how many men would be required to dig a ditch 350 yards long, 3 feet wide, and $1\frac{1}{2}$ deep, in 3 days of 8 working hours? Now in this, and all other examples of Compound Proportion, it must be noted that there is an odd number of terms—in this case 11—given; that one of these terms—in this case 8 men—is of the same kind as the answer, and that the others may be arranged in pairs of terms of a like kind, one of each pair being contained in the supposition, and the other in the demand. Each pair must have its influence in altering the value of the answer, and hence the term like the answer must be multiplied by the fractions formed by the pairs of terms, the term in the *demand* being numerator where the proportion is *direct*, and the term in the *supposition* being numerator where the proportion is *inverse.* In the question just proposed, we first consider that the number of men is directly proportional to the length, width, and depth of the ditch, and inversely proportional to the

number of days and the number of working hours in each day. The answer will then be—

$$8 \text{ men} \times \frac{350 \text{ yds.}}{100 \text{ yds.}} \times \frac{3 \text{ ft.}}{2 \text{ ft.}} \times \frac{1\frac{1}{2} \text{ ft.}}{1\frac{1}{4} \text{ ft.}} \times \frac{1 \text{ day}}{3 \text{ days}} \times \frac{10 \text{ hrs.}}{8 \text{ hrs.}} =$$

$$\overset{\text{men}}{\frac{8}{1}} \times \frac{350}{100} \times \frac{3}{2} \times \frac{6}{5} \times \frac{1}{3} \times \frac{10}{8} = 21 \text{ men}.$$

If the 4 lb. loaf costs $7\frac{1}{2}d.$ when wheat is 50 shillings per quarter, what should be paid per cwt. for bread when wheat is 60 shillings per quarter?

Here the term like the answer is $7\frac{1}{2}d.$, and price is directly proportional to the weight bought, and the price of wheat.

Hence answer $= 7\frac{1}{2}d. \times \frac{112 \text{ lb.}}{4 \text{ lb.}} \times \frac{60 \text{ shillings}}{50 \text{ shillings}} = \frac{5s.}{8} \times \frac{112}{4} \times \frac{60}{50} = £1\ 1s.$

If 25 men earn £37 10*s.* in 5 days, how long will it take 3 men to earn £7 4*s.*?

Here the term like the answer is 5 days, and the time of earning is directly proportional to the amount to be earned, and inversely proportional to the number of men.

Hence answer $= 5 \text{ days} \times \frac{£7\ 4s.}{£37\ 10s.} \times \frac{25}{3} = \frac{\text{days}}{1} \times \frac{144}{750} \times \frac{25}{3} = 8 \text{ days}.$

A question in Compound Proportion is equivalent to two or more questions in Single Rule of Three. Thus the last example might have been separated into two, and worked thus:—

If 25 men earn £37 10*s.* in five days, how long will they take to earn £7 4*s.*?

Here answer $= 5 \text{ days} \times \frac{£7\ 4s.}{£37\ 10s.} = \frac{5 \text{ days}}{1} \times \frac{144}{750} = \frac{24}{25} \text{day}.$

Again. If 25 men earn £7 4*s.* in $\frac{24}{25}$ day, how long will 3 men take to earn the same amount?

Here answer $= \frac{24}{25} \text{ day} \times \frac{25 \text{ men}}{3 \text{ men}} = \frac{24}{25} \text{ day} \times \frac{25}{3} = 8$ days as before.

And as an example of Compound Proportion is merely a combination of several Single Rule of Three questions, the rule that has been given as sometimes practically useful for these may be modified so as to suit the more complicated

case, being as follows: *Increase or diminish the term like the answer in the ratios indicated by the other pairs of terms,* the question whether there should be increase or diminution being settled as if the term like the answer and any pair constituted the three terms of a Single Rule of Three question.

Thus. If the carpet of a room cost £3 12*s.*, the carpet being $\frac{3}{4}$ yd. wide and 2*s.* 3*d.* a yard; what would it cost if it were 1 yd. wide and 3*s.* 6*d.* a yard? Here the cost will be less than £3 12*s.* in consequence of the increased width, and more than £3 12*s.* in consequence of the increased price.

$$\text{Hence answer} = £3\ 12s. \times \frac{\frac{3}{4}\text{ yd.}}{1\text{ yd.}} \times \frac{3s.\ 6d.}{2s.\ 3d.} = \frac{£18}{5} \times \frac{3}{4} \times \frac{14}{9} = £4\ 4s.$$

A man travels 600 miles in 27 days when the day is $13\frac{1}{3}$ hours long; how many days will he require to travel 370 miles when the day is 12 hours long? Here the number of days will be less than 27 on account of the decreased distance, and more than 27 on account of the diminished length of the day.

Hence answer =

$$27\text{ days} \times \frac{370\text{ miles}}{600\text{ miles}} \times \frac{13\frac{1}{3}\text{ hours}}{12\text{ hours}} = \frac{27}{1}\text{ days} \times \frac{370}{600} \times \frac{40}{36} = 18\frac{1}{2}\text{ days.}$$

As was stated before, this method of treating questions in Proportion is inferior to that previously given. It may be used, however, with advantage as a check upon the accuracy with which the former method has been applied.

Ex. 32.

1. If 18 men can do a certain piece of work in 12 days of 8 hours, how many men will be required to do as much in 24 days of 9 hours?

2. If 25 horses can be kept 13 days for £32 10*s.*, how many can be kept 6 weeks for 20 guineas?

3. A tradesman reduces the price of a number of articles from 7*s.* to 6*s.* and thereby sells 8 in the time he formerly sold 5. The cost price being 4*s.* and his gains under the old system £3 10*s.* a week, how much would he gain in 50 days after the alteration?

4. If on a tour of 8 weeks a party of 6 spend £228, how much would be spent by 10 persons on a tour of 12 weeks in a country where travelling is only three-fourths as expensive?

5. If 23 tons 8 cwt. 2 qr. 8 lb. can be carried 196 miles for £19 2*s.* 8*d.*, ow much should be charged for taking 27 tons 2 cwt. 150 miles?

6. If 2 clerks working $7\frac{1}{2}$ hours a day can copy 630 pages of 35 lines ach in 14 days, how many lines must there be in a page if 10 clerks orking 9 hours a day can copy 7560 pages in 24 days?

7. If provisions for 155 persons for 48 days cost £325 10*s.*, how long ill £173 5*s.* maintain 132 persons?

8. The carpet of a room being 27 inches wide and 3*s.* 8*d.* a yard, osts £11 11*s.* Find the cost of a carpet 36 inches wide, at 3*s.* 4*d.* yard.

9. If the 4 lb. loaf cost 7*d.* when wheat is 42*s.* the quarter, what ould be the weight of a twopenny loaf when wheat is 48*s.* the uarter?

10. Supposing that 5 colonial labourers are equivalent to 4 English, hen working for the same time, and that the former are paid 10*s.* per ay of 7 working hours, while the latter receive 4*s.* 6*d.* per day of 10 ours; what would be paid in the colony for work which would cost 346 10*s.* in England?

11. If in walking 10 miles in 3 hours, 11 steps are taken in 6 seconds, ow many steps $\frac{7}{8}$ as long must be taken per minute in order to walk 4 miles in $5\frac{1}{2}$ hours?

12. A man completes a journey of 540 miles in 15 days, travelling hours a day. In how many days would he perform a journey of 945 iles, travelling 10 hours a day, at the same rate?

IISCELLANEOUS EXAMPLES ON CHAPTERS III. IV. V.

1. Find the L. C. M. of 18, 22, 110, 3, 72, 88; and the G. C. M. of 489 and 46407.

2. Find the values of (i.) $\frac{1}{2}+\frac{3}{7}+2\frac{5}{12}+3\frac{7}{13}$; (ii.) $3\frac{1}{6}-4\frac{1}{5}+10-7\frac{1}{12}+5\frac{1}{2}$; (iii.) $\dfrac{3\frac{1}{2}-2\frac{1}{3}}{8\frac{1}{12}+4\frac{1}{6}}$; (iv.) $\frac{7}{8}$ of $2\frac{1}{9}$ of $\frac{16}{19}$ of $\frac{3}{14}$ of £1 4*s.*

3. Reduce to their lowest terms $\frac{3098}{3600}$ and $\frac{1557}{1903}$.

4. Find the fractions equivalent to the decimals ·0375, 2·032, and 5135.

5. Reduce to decimals $\frac{13}{16}$, $\frac{17}{592}$, and find the values of ·8 ÷ ·04; 80 ÷ 04; ·008 ÷ 40; ·008 ÷ ·004.

6. Reduce £3 15*s.* $6\frac{1}{4}$*d.* to the fraction of £5, and find the value of $\frac{3}{8}$ of 17 cwt. 3 qr. 8 lb. 12 oz.

7. If a steam-boat move 30 feet forward for each revolution of the addle wheels, how many revolutions per minute will be required for it o go 270 miles in 24 hours?

8. What is the cost of 43 yards 2 feet 6 inches of oilcloth at 4*s.* 6*d.* per yard?

9. Find the G. C. M. of 18315 and 26730, and the L. C. M. of 18, 24, 192, 72, 60.

10. Add together $2\frac{1}{9}$, 4, $\frac{3}{7}$, $6\frac{1}{12}$, $1\frac{11}{14}$, and from this sum subtract $10\frac{47}{48}$.

11. Reduce $3\frac{11}{25}$, $11\frac{1}{333}$ to decimals, and 1·1875, $\cdot 6\dot{2}9\dot{7}$ to fractions.

12. Reduce $\frac{182952}{249480}$ to its lowest terms.

13. Multiply 19·23 by ·0024 and divide the product by 720.

14. Find what fraction 2*s.* $3\frac{1}{2}$*d.* is of £10, and find the value of $3\frac{4}{17}$ of 1 ton 18 cwt. 1 qr.

15. Express $\frac{3}{4}$*d.* as the decimal of £100, and find the value of ·7825 of 12*s.* 6*d.*

16. What quantity of oats at 3*s.* a bushel would pay for half a mile of draining pipes, each pipe being 16 inches in length, and the price £1 5*s.* per 100?

17. After paying a tax of 7*d.* in the £ on his income, a gentleman lays by $\frac{2}{5}$ of the remainder and spends £5 16*s.* 6*d.* a week. What is his gross annual income? (1 year = 52 weeks.)

18. Find the value of 6 acres 2 roods 20 perches at £2 5*s.* 6*d.* per acre.

19. If by working 9 hours a day 8 men can earn £450 in 25 weeks, how many men must work 8 hours to earn £120 in 4 weeks?

20. Find the G. C. M. of 26400 and 5104, and the L. C. M. of 24, 15, 8, 45, 36.

21. Reduce to their lowest terms $\frac{475}{1900}$, $\frac{2048}{2250}$, and $\frac{58764}{80004}$.

22. Find by Practice the value of 3 tons 5 cwt. 2 qr. 12 lb. at £16 16*s.* per ton, and obtain the same result by Rule of Three.

23. Add together $4\frac{2}{7}$, $3\frac{1}{5}$, $\frac{13}{49}$, $6\frac{1}{8}$, $\frac{11}{196}$, and reduce to fractions $4\cdot 9\dot{3}$, $\cdot 17\cdot 10\dot{6}03\dot{9}$.

24. A father being 38 years of age and his son 6, how long will it be before the son is half the age of his father?

25. A grandfather is 78 and his grandson 18 years of age. Find how long ago the grandfather was 5 times as old as his grandson, and also what was the age of the latter when the grandfather was 877 times as old.

26. A winding road up a mountain has a uniform rise of 220 feet in a mile, and the cost of making it is 4*s.* 6*d.* per linear yard; what would be the expense of carrying it to a height of 2000 feet?

27. Find the value of $2\frac{13}{17}$ of 23 bushels 1 peck 1 gallon, and reduce 8 dwt. 12 gr. to the fraction of 3 lb. troy.

28. Find the area of a room 18 ft. 9 in. long, and 13 ft. 9 in. wide;

the number of yards of carpet 2 ft. 1 in. in width necessary to cover it, and the cost of the carpet at 2*s.* 8*d.* per yard.

29. Find the value of ·08375 of 7 cwt. and reduce 16 lb. 10 oz. to the decimal of a ton.

30. Multiply $14{\cdot}173\dot{7}804\dot{8}$ by ${\cdot}\dot{0}08819\dot{1}$.

31. A rod of brickwork being $272\frac{1}{4}$ square yards 14 inches thick, how many bricks 9 inches long, $4\frac{1}{2}$ inches broad, and $2\frac{1}{2}$ inches thick would it contain, supposing that the mortar occupied $\frac{1}{16}$ of the whole space of the work?

32. The gas burned in a house between Sep. 29 and Dec. 31 costs 19*s.* $4\frac{1}{2}$*d.* at 4*s.* 2*d.* per 1000 feet. What is the average daily consumption?

33. A father leaves $\frac{1}{2}$ his money to his eldest son, $\frac{2}{3}$ as much to his second son, and £500 each to his two daughters. What does he leave altogether?

34. What will be the cost of $\dfrac{\frac{2}{3}-{\cdot}025}{\frac{11}{40}+2{\cdot}0625}$ of $1{\cdot}\dot{0}\dot{3}$ of 3 qr. 4 lb. of sugar at $3\frac{1}{2}$*d.* per lb.?

35. What is the value of a silver cup weighing 13 oz. 3 dwt. 18 gr. at 5*s.* 4*d.* per oz.?

36. The carpet for a room 21 ft. 6 in. long and 17 ft. 11 in. wide costs 5 times as much as the paper, and its width is $\frac{10}{11}$, and its cost per yard $12\frac{1}{2}$ times that of the paper. Find the height of the room.

37. Find the value of ${\cdot}\dot{2}8571\dot{4}$ of £3 + ${\cdot}\dot{6}2\dot{9}$ of 9*s.* + $7{\cdot}3\dot{4}\dot{2}$ of 2*s.* 6*d.*, and reduce the result to the decimal of $\frac{62}{77}$ of £5 2*s.* 3*d.*

38. Multiply £14 3*s.* 2*d.* by 2·065, and divide 2 miles 1 furlong 140 yards by 1·2125.

39. There are two competing lines of omnibuses along the same road. The first consists of 20 omnibuses, each making the double journey 5 times in a day, and taking on the average 9*s.* each way. The second consists of 15, going and returning 6 times in a day, and taking an average of 7*s.* 6*d.* The profits of the second line, which are $\frac{1}{5}$ of the total receipts, are $\frac{2}{3}$ of the profits of the first line. What fraction of the receipts of the first line are profits?

40. The ratio of the circumference to the diameter of a circle is equal to sixteen times the infinite series $\frac{1}{5}-\frac{1}{3\times 5^3}+\frac{1}{5\times 5^5}-\frac{1}{7\times 5^7}+$ &c., minus four times the infinite series $\frac{1}{239}-\frac{1}{3\times 239^3}+\frac{1}{5\times 239^5}-$ &c. Find the value of this ratio correct to 6 places of decimals.

41. If 17 acres 3 roods 800 yards cost £72 5*s.* 2*d.*, what will be the price of 2 roods 15 perches?

42. A clock is 5 minutes slow at 10 P.M. Sep. 27, and $4{\cdot}14\dot{8}$ minutes fast at 20 minutes to 7 P.M. Oct. 4. At what time was the clock exactly right, and what would be the true time when the clock indicated 5 hr. 36 m. 5 sec. A.M. on Oct. 6?

43. If ·2875 of a pint of oil is worth 1·509375*d.*, what should be given for 9 gallons?

44. Reduce $\dfrac{{\cdot}0325 \times 1{\cdot}\dot{6}}{{\cdot}3\dot{6}9\dot{3}} \div \dfrac{{\cdot}25 - {\cdot}0173}{1{\cdot}34 + \frac{9}{20}}$ to a decimal.

45. Find the value of ${\cdot}53\dot{0}\dot{3} \times 3{\cdot}\dot{4}2857\dot{1} \times {\cdot}625 \times 2{\cdot}\dot{4}390\dot{2} \times {\cdot}\dot{8}4615\dot{3} \times$ 4·1 of 1 mile 5 fur.

46. Find, without calculating the decimals themselves, the numbers of non-circulating and circulating figures in the decimals equivalent to the following fractions: $\dfrac{3 \text{ lb. troy.}}{11 \text{ lb. av.}}$; $\dfrac{17 \text{ days}}{41 \text{ years}}$; $\dfrac{1 \text{ min. } 21 \text{ sec.}}{14 \text{ wk. } 3 \text{ days}}$; $2\frac{11}{73} \times \frac{319}{424} \times \frac{363}{999}$

47. If 2 tons 3 cwt. 1 qr. $3\frac{1}{2}$ lb. of iron cost £6·925, how much may be bought for £·96?

48. If first, second, and third-class railway fares are in the proportion of 9 : 7 : 4; and 31, 43, and 67 passengers by each class respectively can go 25 miles for £22 1*s.* 8*d.*, how far can 11 first-class, 27 second-class, and 53 third-class go for £31 15*s.* 5*d.*?

49. Find the value of the infinite series $2 + \frac{1}{2} + \dfrac{1}{2 \times 3} + \dfrac{1}{2 \times 3 \times 4} + \dfrac{1}{2 \times 3 \times 4 \times 5} +$ &c. correct to 6 places of decimals.

50. A certain weight less than 27 cwt. when expressed as a decimal of that quantity has 4 decimal places, and when expressed as the decimal of $68\frac{3}{4}$ cwt. has 2, there being in neither case any circulating period. Find what the weight must be.

CHAPTER VI.

INTEREST, STOCKS, AND EXCHANGE.

Interest is the price paid for the use of money for any given time. When interest is proportional both to the amount used, which is called the *Principal*, and the time for which it is used, it is called *Simple Interest*. To find how much interest is due in any particular case, we must therefore know what would be due on some fixed amount for some fixed time, and the question then becomes one of Compound Proportion. This fixed sum of money is by common consent £100, and the fixed time is 1 year, and interest at the rate of £4, £5, or £6 for this amount and this period is shortly termed interest at 4 *per cent.*, 5 *per cent.*, 6 *per cent.* respectively. Hence the question, 'What is the simple interest on £635 at 4 per cent. for 3 years?' is merely another and a shorter form of the question, 'If £4 be paid for the use of £100 for 1 year, how much should be paid for the use of £635 for 3 years?' By using compound proportion we obtain at once:

$$\text{Required Interest} = \frac{4}{1} \times \frac{635}{100} \times \frac{3}{1} = \frac{381}{5} = £76\ 4s.$$

In cases of this kind, where arithmetical principles are applied to determine practical questions of frequent occurrence, it is often convenient to have a general solution applicable to any special instance. Thus if R be considered to represent any rate per cent., T any time expressed in years or fractions of a year, and I the interest on any principal P, then $I = \frac{R}{1} \times \frac{P}{100} \times \frac{T}{1} = \frac{P \times R \times T}{100}$, whence is

derived the ordinary rule for finding simple interest, *Multiply the principal by the rate per cent., and by the number of years, and divide by* 100. In applying this rule it will be found sometimes better to express the quantities as fractions, and sometimes better to use compound multiplication, and to divide by 100, by cutting off two figures towards the right. The principal, when increased by the interest, is called the *amount*, and hence, if this be represented by the letter A, A=P+I. A statement of equality is called an *equation*, and the two equations, $I=\frac{P\times R\times T}{100}$ and $A=P+I$, contain in a concise form the whole theory of simple interest. The following are examples :

Find the interest on £340 12*s.* 6*d.* for $2\frac{1}{3}$ years at $4\frac{1}{2}$ per cent.

$$340\tfrac{5}{8}\times 2\tfrac{1}{3}\times 4\tfrac{1}{2}\times\tfrac{1}{100}=\frac{2725}{8}\times\frac{7}{3}\times\frac{9}{2}\times\frac{1}{100}=\frac{2289}{64}=£35\ 15s.\ 3\tfrac{3}{4}d.$$

Find the interest on £711 1*s.* 4*d.* for $1\frac{3}{4}$ years at 3 per cent.

$$711\tfrac{1}{15}\times 1\tfrac{3}{4}\times\tfrac{3}{1}\times\tfrac{1}{100}=\frac{10666}{15}\times\frac{7}{4}\times\frac{3}{100}=\frac{37331}{1000}=£37\ 6s.\ 7\tfrac{11}{25}d.$$

Find the amount of £815 14*s.* 9*d.* at $3\frac{1}{2}$ per cent. for $2\frac{1}{2}$ years.

£	*s.*	*d.*
815	14	9
		$3\frac{1}{2}$
2447	4	3
407	17	$4\frac{1}{2}$
2855	1	$7\frac{1}{2}$
		$2\frac{1}{2}$
5710	3	3
317	4	$7\frac{1}{2}$
60·27	7	$10\frac{1}{2}$
20		
5·47		
12		
5·74		

	£	*s.*	*d.*
Interest =	£60	5	$5\frac{149}{200}$
Principal =	815	14	9
Amount =	£876	0	$2\frac{149}{200}$

$$\frac{74\frac{1}{2}}{100}=\frac{149}{200}$$

What is the amount of £1000 for 10 days at 5 per cent. per annum?

$$\text{Here interest}=\frac{£1000}{1}\times\frac{10}{365}\times\frac{5}{100}=\frac{£100}{73}=£1\ 7s.\ 4\tfrac{56}{73}d.$$

And therefore amount = £1001 7*s.* $4\frac{56}{73}$*d.*

Find the interest on £531 8*s.* 2*d.* at 5 per cent. from Dec. 31, 1861, o Nov. 1, 1863.

Here the time is 1 year 305 days, or $1\frac{61}{73}$ years. And

$$\frac{5}{1} \times \frac{134}{73} \times \frac{1}{100} = \frac{67}{365}$$

	£	*s.*	*d.*
	531	8	2
	67		
	3717		
s. d.	3186		
(67 × 6 8) . . .	22	6	8
(67 × 1 4) . . .	4	9	4
(67 × 0 2) . . .	0	11	2
	35604	7	2

£	*s.*	*d.*
365)35604	7	2(97 10 $11\frac{31}{365}$
3285		
2754		
2555		
199		
20		
3987		
3650		
337		
12		
4046		
4015		
31		

Interest = £97 10 $11\frac{31}{365}$

Ex. 33.

1. Find the interest on £362 10*s.* for 5 years at 4 per cent.
2. Find the interest on £463 17*s.* 6*d.* for $2\frac{1}{2}$ years at $3\frac{1}{2}$ per cent.
3. Find the amount of £113 15*s.* for 7 years 5 months at $4\frac{1}{4}$ per cent.
4. Find the interest on £225 8*s.* for 9 years 2 months at $6\frac{1}{4}$ per cent.
5. Find the amount of £78 7*s.* 6*d.* for $4\frac{1}{2}$ years at 5 per cent.
6. Find the amount of £1192 for $3\frac{1}{8}$ years at 4 per cent.
7. Find the interest on £739 6*s.* 3*d.* for 5 years 219 days at $3\frac{3}{4}$ per cent.
8. Find the amount of £273 14*s.* $11\frac{1}{2}$*d.* for 3 years 45 days, at 5 per cent.
9. Find the interest on £2000, from Jan. 5 to Feb. 9, at 6 per cent.
10. Find the interest on £815 19*s.* $4\frac{1}{2}$*d.* from Jan. 1, 1857, to Mar. 14, 1860, at $3\frac{1}{8}$ per cent.

Of the five quantities, P, A, R, T, I, any three being given except P, A, I, the rest may be found by ordinary proportion. The reason why the three, P, A, I, may not be given to find the rest is that these three have between themselves the relation P+I=A, making each dependent on the other two, so that what is apparently giving three quantities is in reality only giving two. The pupil who is

acquainted with elementary geometry may call to mind a corresponding exception in the determination of triangles. From any three out of the seven particulars of a triangle—the three sides, the three angles, and the area—the other four may be determined; but to this rule there is one exception. The three angles must not be the particulars given, inasmuch as these have a relation among themselves, their sum being equal to two right angles. Of the five quantities, P, A, R, T, I, three may be chosen in ten different ways, and we will consider these in order. It must, however, be premised that whenever any two of the quantities P, A, I are known, the three may be considered as known. Thus if P R A are given, we may consider that P R A I are given, as I may be found at once by subtracting P from A.

(1) P R T given. This case has already been fully considered.

(2) I R T given. Here $P = £100 \times \frac{\text{Interest}}{\text{Interest on £100 for same time}}$

$= £100 \times \frac{I}{R \times T}$

(3) A R T given. Here $P = £100 \times \frac{\text{Amount}}{\text{Amount of £100 in same time}}$

$= £100 \times \frac{A}{100 + (R \times T)}$

(4) (5) (6) P I R, or P A R, or A I R. Here $T = \frac{\text{Interest}}{\text{Interest for 1 year}}$

$= I \div \frac{P \times R}{100}$

(7) (8) (9) P I T, or P A T, or A I T. Here $R = \frac{\text{Interest}}{\text{Interest at 1 per cent.}}$

$= I \div \frac{P \times T}{100}$

(10) P A I given. This is the exceptional case from which nothing can be found. The following are examples of these general rules.

Example of (2). What principal will produce £20 5*s.* 6*d.* interest, in $1\frac{1}{4}$ years at 6 per cent.?

Here interest on £100 for same time $= \frac{5}{4} \times \frac{6}{1}$, therefore required principal $= \frac{100}{1} \times 20\frac{11}{40} \times \frac{4}{5} \times \frac{1}{6} = \frac{100}{1} \times \frac{811}{40} \times \frac{4}{5} \times \frac{1}{6} = \frac{811}{3} = £270\ 6s.\ 8d.$

Example of (3). What principal would amount to £873 16*s.* 6*d.* in $5\frac{1}{4}$ years at $3\frac{3}{4}$ per cent.?

Here amount of £100 in same time $=100+\left(\frac{16}{3}\times\frac{15}{4}\right)=$ £120, therefore required principal $=\frac{100}{1}\times 873\frac{33}{40}\times\frac{1}{120}=\frac{100}{1}\times\frac{34953}{40}\times\frac{1}{120}=\frac{11651}{16}$ $=$ £728 3*s.* 9*d.*

Example of (5). In what time would £650 amount to £734 10*s.* at 3 per cent.?

Here interest for 1 year $=650\times\frac{3}{100}$ therefore required time

$$=84\frac{1}{2}\times\frac{1}{650}\times\frac{100}{3}=\frac{169}{2}\times\frac{1}{650}\times\frac{100}{3}=\frac{13}{3}=4\frac{1}{3}\text{ years.}$$

Example of (9). At what rate per cent. would a certain principal in $2\frac{1}{3}$ years produce £58 16*s.* interest, and consequently increase to £618 16*s.*

Here the principal must be £560, and the interest at 1 per cent. would be $560\times\frac{7}{3}\times\frac{1}{100}$, therefore required rate per cent. $=58\frac{4}{5}\times\frac{1}{560}\times\frac{3}{7}\times\frac{100}{1}$ $=\frac{294}{5}\times\frac{1}{560}\times\frac{3}{7}\times\frac{100}{1}=\frac{9}{2}=4\frac{1}{2}.$

Ex. 34.

1. In what time would £800 increase to £1020 at 6 per cent.?

2. At what rate per cent. would £763 2*s.* 6*d.* produce £133 10*s.* $11\frac{1}{4}$*d.* interest in 5 years?

3. A certain principal, invested at $7\frac{1}{2}$ per cent., produces £251 1*s.* 3*d.* interest, and consequently increases to £1255 6*s.* 3*d.* Find the time.

4. What principal would in 3 years, at 4 per cent., produce £68 6*s.* 3*d.* interest?

5. At what rate per cent. would £115 4*s.* increase to £154 16*s.* in $6\frac{1}{4}$ years?

6. In what time would £983 10*s.* $3\frac{3}{4}$*d.* produce £78 13*s.* $7\frac{1}{2}$*d.* interest at 4 per cent.?

7. The interest being £184 4*s.* $4\frac{1}{2}$*d.*, the amount £866 10*s.* $2\frac{1}{2}$*d.*, and the time $4\frac{1}{2}$ years, find the rate per cent.

8. What principal would in $1\frac{1}{2}$ years, at 3 per cent., increase to £1996 14*s.* $8\frac{1}{10}$*d.*

9. For how many years and days would £17 17*s.* 6*d.* be the interest on £86 19*s.* 10*d.* at 5 per cent.?

10. At what rate per cent. would £53 8*s.* 9*d.* produce £13 1*s.* 3*d.* interest in $2\frac{2}{3}$ years?

If a sum of money be due at some future time, and it be required to determine the worth of that debt now, or the *present value*, as it is called, the question is one belonging to case (3). The difference between the present value and the amount due is called the *Discount.* For the amount due, the present value, the time, the rate per cent., and the discount, put the letters A P T R D. Then from case (3) we have $P=100\times\frac{A}{100+(R\times T)}$ or $P=A\times\frac{100}{100+(R\times T)}$. Also from our definition of discount $D=A-P$. These two equations contain in themselves the whole theory of discount.

If we wish to find D without previously finding P, and subtracting, we must proceed in the following manner:— In a certain time, T, £100 would have amounted to £100+(R × T). Hence the present value of £100+(R × T) due T years hence, is £100, and the discount is by subtraction R × T. Consequently the discount on any other sum of money, A, must be a proportional amount, and hence $D=R\times T\times\frac{A}{100+(R\times T)}$, or $D=A\times\frac{R\times T}{100+(R\times T)}$. The above equations lead to the following rules :—To find the present value, *diminish the amount due in the ratio of £100 to £100 together with the product of the rate and time.* To find the discount, either find the present value, and subtract it from the amount due, or *multiply the amount due by the ratio of the product of the rate and time to £100 together with that product.*

As an example, let it be required to find the present value of £375 7*s*. 6*d*, due 3 months hence, at 6 per cent.

$$\text{Here } 6\times\frac{1}{4}=\frac{3}{2}=1\tfrac{1}{2},\text{ and present value}=375\tfrac{3}{8}\times\frac{100}{101\frac{1}{2}}$$

$$=\frac{3003}{8}\times\frac{200}{203}=\frac{10725}{29}=£369\ 16s.\ 6\tfrac{18}{29}d.$$

Next, to find the discount on £546 13*s*. 4*d*. for 6 months, at 5 per cent. Here we may either find the present value and subtract, or proceed thus:

$$5\times\tfrac{1}{2}=2\tfrac{1}{2}\quad 546\tfrac{2}{3}\times\frac{2\frac{1}{2}}{102\frac{1}{2}}=\frac{1640}{3}\times\frac{5}{205}=\frac{40}{3}=£13\ 6s.\ 8d.$$

The *interest* on £546 13*s*. 4*d*. for 6 months, at 5 per cent., would be

$$546\tfrac{2}{3} \times \tfrac{5}{1} \times \tfrac{1}{2} \times \tfrac{1}{100} = \tfrac{1640}{3} \times \tfrac{5}{1} \times \tfrac{1}{2} \times \tfrac{1}{100} = \tfrac{41}{3} = £13\ 13s.\ 4d.$$

It appears, therefore, that in this particular case the discount is less than, but nearly equal to, the interest, the difference between the two being 6*s*. 8*d*. For short periods, less than a year, the difference between interest and discount is trifling, and it is for short periods generally that discount has to be calculated for practical purposes. Hence it is found convenient by merchants, bankers, and others, to substitute interest for discount in their commercial transactions. The effect of this substitution is to obtain, at the expense of slight inaccuracy, greatly increased facilities of calculation. Interest is more easily and quickly calculated than discount, and although for the purposes of merchants and bankers, these calculations are effected by means of tables which give the required amount at a glance, there are, nevertheless, two important advantages gained by the substitution—firstly, that instead of separate tables for interest and discount, one set for interest does all the work; and, secondly, that the interest tables are more compact and more practically applicable than the discount tables. The latter advantage arises from the fact that interest is directly proportional to each of the three—principal, time, and rate per cent.; while discount is proportional to the amount due, but is not proportional to time nor to rate per cent. Interest is used in the place of discount so extensively, that probably there is no instance of the employment of the latter. As an arithmetical exercise it is, however, useful to be able to calculate it, and questions wherein it is required are constantly set in examination papers. To prevent confusion, throughout the remainder of this book, the exact discount, which is found according to the principles that have been laid down, and which is not used in common life, will be called the *true discount*, and the

amount commonly deducted from sums due at a future time will be called *discount.*

Of the various kinds of transactions in which discount is employed, it is only necessary to mention one as possessing any special peculiarity. That is the case of bills. A bill is a commercial document, drawn on a certain day, and undertaking the payment of a certain sum in a given time. This may be converted into ready money, or *discounted,* as it is termed, at any time before it becomes payable, and discount will then be deducted for the time intervening. The calculation of this deduction is merely an instance of the application of the rule for finding simple interest, but the following commercial regulations must be attended to:—

Three *days of grace* are allowed, so that a bill drawn on May 14th, for three months, becomes legally due on August 17th. Also, if the bill would be nominally due on some impossible date, as the 30th or 31st of February, or the 31st of September, it is considered nominally due on the last day of the month, and therefore legally due on the 3rd of the following month. If a bill would legally fall due on a Sunday, it is, in Great Britain, payable on the Saturday; and in Ireland on the Monday.

Find the discount on a bill for £500, drawn March 13th, at 3 months, and discounted April 5th, at 6 per cent.

Here from April 5th to June 16th there are 72 days; therefore discount=

$$\frac{500}{1}\times\frac{6}{1}\times\frac{72}{365}\times\frac{1}{100}=\frac{432}{73}=£5\ 18s.\ 4\tfrac{20}{73}d.$$

The following particulars with regard to true discount may serve to explain its nature more clearly:—

1. The common practice of charging interest instead of true discount, when bills are converted into money, deviates from accuracy in this one respect, that a loan is made, and the interest on that loan is asked for at once, instead of being demanded when it has accrued in the regular course of time. Thus, if a merchant take a bill at three months for £100, and give cash for it, he virtually lends £100 for

three months, and at the expiration of that time he might fairly demand three months' interest. But, instead of waiting, he makes the charge at once, and thereby obtains a little more than his due. What he ought to charge would obviously be the true present value of the interest. Hence the following relations hold good with regard to the interest and true discount on any sum :—

True Discount=true present value of interest.

Interest=the amount of the true discount for given time.

Difference between true discount and interest=either the true discount on the interest, or the interest on the true discount.

To illustrate these points by an example, we will take the case of £410 at 5 per cent. for 6 months. Here the interest is £10 5*s*., and the true discount is £10; and we find that for the given time, and at the given rate, £10 is the true present value of £10 5*s*., and £10 5*s*. is the amount of £10, while the difference, 5*s*., between the interest and true discount is the true discount on £10 5*s*., and is also the interest on £10.

2. A curious relation exists between the fractions which respectively indicate the ratios that the interest and true discount bear to the sum on which they are calculated. Expressed generally, these fractions are $\frac{R \times T}{100}$ and $\frac{R \times T}{100 + (R \times T)}$. Now it may be noticed that in these the numerators are the same, while the denominator of the second is the sum of the numerator and denominator of the first. We may, from this relation, at once deduce true discount from interest, or interest from true discount. Thus, suppose that the interest on any sum of money were $\frac{3}{49}$ of that sum, then the true discount must be $\frac{3}{52}$ of the same. Again, true discount being $\frac{6}{125}$, interest would be $\frac{6}{119}$ of the amount on which either was charged.

Ex. 35.

1. Find the true discount on £275 8*s*. 9*d*. for 1 year at 5 per cent.
2. Find the true discount on £54 3*s*. 1*d*. for 6 months at 5 per cent.
3. Find the true present value of £420 12*s*. 3*d*., due 9 months hence, at 6 per cent.

4. How much per cent., upon the amount due, is the difference between true and common discount in the case of bills at 3 months, interest being 6 per cent.?

5. Find the discount on a bill for £400 drawn May 9th, at 3 months, and discounted June 12th, at 8 per cent.

6. Find the discount on a bill for £69 7*s*., drawn June 14th, at 90 days, and discounted July 27th, at 5 per cent.

7. Find the discount on a bill for £580, drawn March 4th, at 6 months, and discounted April 14th, at $7\frac{1}{2}$ per cent.

8. At what rate per cent. would the true discount for 5 months be $\frac{1}{49}$ of the amount due?

9. If the interest on £263 15*s*. for a certain time be £14 10*s*. $1\frac{1}{2}$*d*., by how much is that greater than the true discount for the same time?

10. If the true discount and the interest, both at 4 per cent., are in the ratio of 25 to 27, find the time.

When a sum of money is put out at interest, the interest itself may, as it becomes due, be employed in the same manner. Where this is done, the interest being added to the principal at stated intervals of time, the money is said to be put out at *Compound Interest.* For the present we shall confine our attention to cases where the interval is one year, that being the general custom. The method of calculation usually given is by successive applications of the rule for finding simple interest. Thus, to find the compound interest on £600 for 3 years at 5 per cent.:—Here interest for 1st year$=\frac{600}{1}\times\frac{5}{100}=$£30, and adding this to the principal, there results £630 as the amount on which interest will be paid in the second year. Hence interest for second year$=\frac{630}{1}\times\frac{5}{100}=$£31 10*s*., and the amount at the end of the second year will be £661 10*s*. Lastly, third year's interest$=661\frac{1}{2}\times\frac{5}{100}=$£33 1*s*. 6*d*., and the final amount will be £694 11*s*. 6*d*., thus making the compound interest received £694 11*s*. 6*d*., *minus* £600, or £94 11*s*. 6*d*. Again, let it be required to find the amount of £735 10*s*. 2*d*. at $3\frac{2}{3}$ per cent. compound interest, for 4 years. To find this exactly to the precise fraction of a penny, would require many figures and much labour. It will, therefore, be only approximately worked out.

£	s.	d.
735	10	2
		11
3)8090	11	10
26·96	17	3·3̇
20		
19·37		
12		
4·473		

£	s.	d.
735	10	2
26	19	4·5
762	9	6·5
		11
3)8387	4	11·5
27·95	14	11·8
20		
19·14		
12		
1·798		

£	s.	d.
762	9	6·5
27	19	1·8
790	8	8·3
		11
3)8694	15	7·3
28·98	5	2·4
20		
19·65		
12		
7·824		

£	s.	d.
790	8	8·3
28	19	7·8
819	8	4·1
		11
3)9013	11	9·1
30·04	10	7
20		
90		
12		
10·87		

£819	8	4·1
30	0	10·9
£849	9	3

Amount at end of 4 years = £849 9*s.* 3*d.*

From this example it will be seen how long and tedious the calculation of compound interest would be where the time was several years. In the numerous cases in which such calculations are required, they are generally performed by the aid of tables. Compound Interest is proportional to the principal, but not to the time nor to the rate per cent., so that the tables must give the compound interest on some given sum of money, generally £1 or £100, for all the usually required periods, at the usual rates per cent. Calculations of compound interest may be made rapidly and easily without these tables, by the aid of a powerful assistant to computation, called *Logarithms*. A brief explanation of the nature and use of logarithms will be given in the next chapter. In order, however, to apply them to compound interest, a somewhat different way of viewing the question must be employed. To illustrate this, we will again refer to the two examples already worked out. First, to find the compound interest on £600 for 3 years, at 5 per cent. At the end of the first year, the interest being $\frac{5}{100}$ of the principal, it is clear that the amount will be greater than the principal in the ratio of 105 to 100. The amount at the end of the first year becomes a new principal for the second, and increases throughout that year in like

manner. Therefore, at the end of the second year, the amount must be $600 \times \frac{105}{100} \times \frac{105}{100}$, and at the end of the third year the amount is $600 \times \frac{105}{100} \times \frac{105}{100} \times \frac{105}{100}$. Actually working this out, we obtain $\frac{600}{1} \times \frac{21}{20} \times \frac{21}{20} \times \frac{21}{20} = \frac{27783}{40} =$ £694 11*s.* 6*d.* as we found before. In general, P being the principal, the effect of the addition of each year's interest is to multiply P by the fraction $\frac{100+R}{100}$. And this is done as many times as there are years, and hence P is altogether multiplied by $\frac{100+R}{100}$ raised to the Tth power, or if A=amount, $A = P \times \left(\frac{100+R}{100}\right)^T$. Applying this equation to the second example just worked, namely, to find the amount of £735 10*s.* 2*d.* at $3\frac{2}{3}$ per cent. compound interest, for 4 years, we have £735 10*s.* 2*d.* = £$735 \cdot 508\dot{3}$; $\frac{100+3\frac{2}{3}}{100} = \frac{311}{300}$, and hence Amount = £$735 \cdot 508\dot{3} \times \left(\frac{311}{300}\right)^4$. By the aid of logarithms the value of this expression might be easily found in two minutes.

The present value of any amount due may be calculated upon the supposition of compound interest as well as in the former manner, thus giving a different kind of present value from that formerly found, and by subtraction from the amount due, giving a different kind of discount. Such a present value would be amount × £100 ÷ amount of £100 in given time at compound interest.

Thus, to find the present value of £167 4*s.* 3*d.*, due 3 years hence, at 5 per cent. compound interest. In 3 years £100 would amount to £115 15*s.* 3*d.*, and therefore required present value $= 167\frac{51}{240} \times 100 \div 115\frac{61}{80}$ $= \frac{40131}{240} \times \frac{100}{1} \times \frac{80}{9261} = \frac{1300}{9} =$ £144 8*s.* $10\frac{2}{3}$*d.*

The general solution of questions of this kind would be that since $A = P \times \left(\frac{100+R}{100}\right)^T$ therefore $P = A \div \left(\frac{100+R}{100}\right)^T$,

or, which is the same thing, $P=A\times\left(\frac{100}{100+R}\right)^T$. The value of this expression in any particular case is at once obtained by logarithms. One great difference between this mode of estimating present value and that depending upon simple interest is, that whereas the latter is perhaps never used, the former is constantly employed in numerous kinds of transactions. In insurance, annuities, reversions, and many other instances, such questions continually occur, and may be found from special tables, or easily calculated by logarithms.

Although, as already mentioned, the interest is added to the principal, generally at intervals of one year, any other interval might be chosen, as six months, one month, a week, or a day. The general statement of a question in compound interest is, *interest being added to the principal at given intervals, find the amount at the end of any given time.* It is a question not unfrequently given as an exercise, and which can easily be determined, though not by mere arithmetic, what would be the amount where interest was supposed to be added at infinitely small intervals of time; that is to say, at intervals shorter than any we can assign, less, for example, than the thousand millionth part of a second.

Find the compound interest on £251 0*s.* 10*d.* for 2 years at 5 per cent. per annum, the interest being added every half-year.

Here the number of intervals in the given time is 4, and the interest for one interval is $2\frac{1}{2}$ per cent. Recollecting that $2\frac{1}{2}$ per cent. is the same as $\frac{1}{40}$, and being contented with a very near approximation to actual exactness, the work will stand as follows:—

	£	*s.*	*d.*		£	*s.*	*d.*
40)	251	0	10	40)	263	15	0·16
	6	5	6·25		6	11	10·50
40)	257	6	4·25	40)	270	6	10·66
	6	8	7·91		6	15	2·07
	263	15	0·16		277	2	0·73

The amount is £277 2*s.* 0·73*d.*, and consequently the compound interest is £26 1*s.* 2·73*d.*

Here the actually exact value is £277 2*s*. 0·72665915625*d*., so that the approximation just arrived at is true to between $\frac{1}{299}$ and $\frac{1}{300}$ of a penny.

If we wished to put the question in a form adapted for calculation by logarithms, we should have R=$2\frac{1}{2}$, and T=4, and therefore the amount=£251 0*s*. 10*d*. $\times \left(\frac{102\frac{1}{2}}{100}\right)^4$, or £251 0*s*. 10*d*. $\times (\frac{41}{40})^4$.

One minor point remains to be noticed. The given time might contain some number of intervals and part of an interval, and in such a case, after calculating the amount up to the end of the last complete interval, the remaining interest must be calculated by the rule for simple interest.

Thus, to find the amount of £120 at 4 per cent. per annum, compound interest, for 2 years and 5 months. Here £120 at 4 per cent. compound interest, for 2 years, would amount to £129 15*s*. $10\frac{2}{25}$*d*., and the amount of this at simple interest for 5 months would be £131 19*s*. $1\frac{31}{125}$*d*. Hence total interest is £11 19*s*. $1\frac{31}{125}$*d*. Note K.

Ex. 36.

1. Find the amount of £1024 for 3 years, at $6\frac{1}{4}$ per cent. compound interest.

2. Find the compound interest on £1627 12*s*. 1*d*. for 4 years, at 4 per cent.

3. Find the amount of £600 at 6 per cent. for 3 years, compound interest.

4. Find approximately the compound interest on £150 for 6 years, at 5 per cent.

5. What is the present value of £507, due 2 years hence, at 4 per cent. compound interest.

6. The amount at the end of 2 years at compound interest being £1478 15*s*., and at the end of 3 years being £1537 18*s*., find the original principal.

7. Find the amount of £800 for 2 years, at 5 per cent. per annum compound interest, the interest being reckoned half-yearly.

8. Which would in 3 years produce the greatest amount, compound interest at 6, or simple interest at $6\frac{1}{2}$ per cent., and what would be the difference on a principal of £1000?

9. Find the amount of £630 at $3\frac{1}{3}$ per cent. per annum, compound interest, for $2\frac{1}{2}$ years.

10. The compound interest at 5 per cent. for 2 years is greater than the simple interest for the same time by 7*s*. 8*d*. Find the principal.

As money unemployed is unproductive, always remaining the same in amount, yielding no profit and receiving no increase, its owners are generally anxious either to apply it to profitable use themselves, or to lend it to others willing to pay them for the privilege of doing so. They may, for this purpose, entrust it to traders or trading companies, or they may lend it to their own or to a foreign government. Just as the prices of other commodities vary, so does the price paid for the use of money; but probably there is nothing else in which the variation of price is regulated by such numerous and complicated causes. It is universally found that the best way of preventing inconvenience from a perpetually varying scale of prices is to establish a free, open, and comprehensive market. And whilst there are, accordingly, in London and other great cities, markets for corn, coals, wool, cotton, &c., so also is there a money market. In London this is established at the Stock Exchange, and the prices paid by governments or public companies for the use of money are published in the newspapers daily, and in the City many times in the course of a day. It would, however, in practice be very difficult to adopt the apparently simplest mode of expressing the fluctuations in the price, namely, changing the rate of interest, for the changes would in this way be represented by small and exceedingly unmanageable fractions. But the same object is effected by a much more delicate method, namely that of stating the corresponding variations in the principal or sum lent. The questions given in arithmetical books under the head of 'Stocks,' whether they have reference to governments, railways, mines, or companies, are all based upon one principle, *that a certain rate of interest is paid upon a nominal sum of money, which is worth a perpetually varying amount of actual money.*

Of these facilities for *investing* money, as it is termed, the Public Funds are the most important. In the reign of William and Mary money was lent to the Government, and

the amount, which was at first inconsiderable, has been since increased from time to time, until the *National Debt* has now reached the sum of about eight hundred millions. This borrowed money has been all spent; it cannot now be employed in a productive manner, but the Government is, nevertheless, liable for the amount, and instead of paying it, provides the annual interest sufficient to pay those persons who choose to become for a time holders of a portion of the debt. This annual interest is supplied by taxation, and so far as taxes are imposed for this object, they are the price paid by the public to ensure the good faith of the Government towards the fundholders—a burden which the public are called upon to bear, inasmuch as it was by the expenditure of this money that England was preserved through many critical periods of history, to enjoy its present prosperity, in which they have a direct interest.

In the case of the Funds, the *nominal sum of money* is £100, and this may be worth any actual sum, more or less. The rate of interest is in most instances 3 per cent., and less commonly $2\frac{1}{2}$, $3\frac{1}{4}$, and 5. According to the different rates of interest, and the time when the debts bearing that interest originated, different names are given to different descriptions of *Stock*, as it is called. Thus, there are the 'Consolidated Annuities,' or 'Consols,' which constitute about one-half of the national debt; there are the 'New Three per cent. Annuities,' the 'Three per cent. Reduced,' 'Bank Stock,' and some others. If Consols are at 95, the meaning is that £95 will buy the nominal £100, on which £3 is paid annually. All questions in stocks are ordinary examples of proportion; but the following rules will be found generally applicable:—

To find money equal to stock. *Multiply by* $\dfrac{\text{Price of stock}}{\pounds 100}$

To find stock equal to money " $\dfrac{\pounds 100}{\text{Price of stock}}$

To find income arising from stock.	*Multiply by*	$\dfrac{\text{Rate per cent.}}{\pounds 100}$
To find income arising from money.	"	$\dfrac{\text{Rate per cent.}}{\text{Price of stock}}$

Before proceeding to give examples, the following particulars with regard to stocks may be mentioned :—

1. The actual rate of interest on money in the Funds is always the lowest that is paid for money at all. The reason is because this kind of investment combines in itself advantages which are not all found together in any other kind. These advantages are three—first, the best possible security, the credit of the nation; secondly, the power of drawing out our money any day we choose; and, thirdly, the certainty that the fluctuations in value of our investment will not be considerable. To these may be added the advantage, that the expenses of effecting or calling in any investment are trifling.

Comparing the Public Funds with other securities, we see that in mortgages chances of fluctuation are avoided, but we cannot draw out our money when we please; and in mines and railways, where we can withdraw our investment at any time, the value of our shares may have increased or decreased in any proportion whatever.

2. Stock is usually bought or sold through the agency of a broker, who finds a seller or a purchaser, as the case may require, and who charges, in the case of the Funds, $\frac{1}{8}$ per cent. commission for his services. Those who are ready in the market, either to buy or sell stock, announce two prices, generally differing by $\frac{1}{8}$, the lower being their buying, and the higher their selling price. Thus, if Consols were quoted at $92\frac{3}{8} \ldots \frac{1}{2}$, you would (taking account of brokerage) realise from £100 stock £92 5*s*., and would have to pay for £100 stock £92 12*s*. 6*d*. In stocks of other descriptions the charge for commission is different, and in the examples in this book, no allowance is to be made for brokerage unless expressly stated.

If Consols are at $93\frac{1}{2}$, what is the actual rate of interest per cent.?

Since £3 is paid upon £$93\frac{1}{2}$, the rate per cent. is $3 \times \frac{100}{93\frac{1}{2}} = \frac{600}{187} = 3\frac{39}{187}$.

What income would be derived from investing £1380 3*s.* 4*d.* in Consols, at 91?

$$\text{Here income} = 1380\tfrac{1}{6} \times \frac{3}{91} = \frac{8281}{6} \times \frac{3}{91} = \frac{91}{2} = £45\ 10s.$$

What sum must be invested in the 5 per cents., at 114, to produce an income of £95?

$$\text{Here required amount} = 114 \times \frac{95}{5} = £2166.$$

Is it more advantageous to invest in the 3 per cents. at 91, or in the $2\frac{1}{2}$ per cents. at 76, and what would be the difference in the incomes arising from investing £1729 in each?

The 3 per cents. yield $\frac{3}{91}$ and the $2\frac{1}{2}$ per cents. $\frac{2\frac{1}{2}}{76}$ of the sum invested. These fractions, $\frac{3}{91}$ and $\frac{5}{152}$, when reduced to a common denominator, are $\frac{456}{13832}$ and $\frac{455}{13832}$. Hence the difference is $\frac{1}{13832}$ of the sum invested, and on £1729 would be $\frac{1729}{13832} = \frac{1}{8} = 2s.\ 6d.$

A man invests £2350 in the 3 per cents., at $93\frac{7}{8}$, and on the price rising to 98, sells out and invests the money in 6 per cent. railway stock, at £144 15*s.* The brokerage for buying and selling the 3 per cents. being $\frac{1}{8}$, and for buying the railway stock $\frac{1}{4}$ per cent., find the difference in his income?

The effect of the brokerage is to make the buying and selling prices of the 3 per cents. 94 and $97\frac{7}{8}$ respectively, and the price of the railway stock £145. Therefore his first income $= £2350 \times \frac{3}{94} = £75$. And his second income $= £2350 \times \frac{100}{94} \times \frac{97\frac{7}{8}}{100} \times \frac{6}{145} = \frac{2350}{1} \times \frac{100}{94} \times \frac{783}{800} \times \frac{6}{145} = \frac{405}{4}$ $= £101\ 5s.$ Therefore difference of income $= £26\ 5s.$

A man invests £457 10*s.* 2*d.* in Consols, at 93. What would he lose by their falling $\frac{1}{2}$ per cent.?

His loss would be £$\frac{1}{2}$ on every £93, and therefore would altogether be

$$457\tfrac{61}{120} \times \frac{\frac{1}{2}}{93} = \frac{54901}{120} \times \frac{1}{186} = \frac{1771}{720} = £2\ 9s.\ 2\tfrac{1}{3}d.$$

Ex. 37.

1. What must be given for £2750 Consols, at $95\frac{1}{2}$, brokerage being $\frac{1}{8}$ per cent.?

2. What amount of Consols, at $97\frac{1}{2}$, could be bought for £1300?

3. What must be the price of Consols when £2724 9*s*. 1*d*. would be given for £2921 13*s*. 4*d*. stock?

4. What income would be derived from investing £4857 15*s*. 10*d*. in the 3 per cents., at 92?

5. What is the price of Consols, when an income of £39 is derived from investing £1197 12*s*. 6*d*. ?

6. The value of a man's 3 per cent. stock is diminished £12 10*s*. by a fall of $\frac{1}{2}$ per cent. What income does he derive from it?

7. Which investment yields the larger income, the 3 per cents. at 97, or the $3\frac{1}{4}$ per cents. at $105\frac{1}{8}$, and on what amount invested would the difference be 15*s*.?

8. What sum must be invested in the $3\frac{1}{2}$ per cents., at 90, to produce an income of 50 guineas?

9. What is the actual rate of interest obtained, when New Zealand 6 per cent. stock is bought at 110?

10. Consols are bought at 92, a half-year's dividend is received, and they are sold at 94. The total increase of capital being £10 10*s*., find the amount invested.

11. On Nov. 20, 1863, London and Westminster Bank shares are worth £$79\frac{7}{8}$, £20 having been paid up on each; Peninsular and Oriental Navigation shares are worth £$84\frac{3}{8}$, £50 having been paid up; and Consols are at $91\frac{1}{8}$. What rates of interest on paid up capital must the Bank and Navigation Company respectively pay, to make the three investments equally profitable?

12. A man invests £645 15*s*. in Consols, and sells out when they have risen $\frac{3}{4}$ per cent., thereby gaining 5 guineas. At what price did he sell?

13. A man invests £551 5*s*. in Consols, at $94\frac{1}{2}$, and afterwards sells out at 96, and buys 5 per cent. railway debentures, at 112. Find the alteration in his income.

14. The holder of certain bank shares, worth £13 5*s*. each, proposes to sell them, and invest in $3\frac{1}{2}$ per cent. stock, at 102. He delays doing so, however, until the shares have fallen to £12 10*s*., and the stock has risen to 104. Some time afterwards, having in the meanwhile received one half-yearly dividend, he sells out at $105\frac{1}{2}$, and calculates that he altogether lost £212 8*s*. 9*d*. by his delay. How many shares had he?

Rates per cent., considered sometimes with reference to the abstract number 100, and sometimes with reference to the concrete number £100, have an immense variety of applications to practical purposes. It would be almost

impossible to furnish a complete list of these, but the following, which are some of the most important, are here given and illustrated by examples :—

1. We have seen that a certain rate per cent. is paid for the use of money, and that a percentage is charged by brokers for their services in buying or selling stock. Similarly, *commission* is charged upon the value of goods bought or sold by agents on account of other persons, and *premium of insurance* is a percentage paid to companies who undertake to pay a certain sum in the event of the death of the insurer, or his suffering loss by a house being burned, or a ship being wrecked. These different kinds of insurances are called *Life*, *Fire*, and *Marine* respectively. The principle may be extended to an endless variety of cases. There are at present societies for insuring against loss to your family from your death or injury by any accident, or by railway accidents only, for providing maintenance and medical attendance for you when ill, for paying funeral expenses, for insuring against loss by dishonest servants, bad debts, hail storms, cattle disease, breakage of plate-glass windows, and, in fact, against almost every kind of destruction of property. All questions connected with these subjects are merely questions of proportion.

What is the premium on a life insurance for £5000, at £3 7*s.* 8*d.* per cent.?

$$\text{Here } 3\tfrac{23}{60} \times \frac{5000}{100} = \frac{203}{60} \times \frac{50}{1} = \frac{1015}{6} = £169\ 3s.\ 4d.$$

What will be the annual cost of insuring a house worth £2000, the premium being 1*s.* 6*d.* and the duty 3*s.* per cent.? What would it have been without the duty?

The effect of the duty is to increase the cost threefold from 1*s.* 6*d.* to 4*s.* 6*d.*, and hence cost of insurance $= £2000 \times \frac{4\frac{1}{2}}{20} \times \frac{1}{100} = 4\tfrac{1}{2} = £4\ 10s.$

Without the duty it would have been $\frac{1}{3}$ as much, namely, £1 10*s.*

A man insures a vessel and cargo, worth £3700, at $7\frac{1}{2}$ per cent. What premium must he pay so as in case of loss to receive the value of the vessel and cargo, and cost of insurance?

Each £100 insured covers £$92\frac{1}{2}$ worth of goods, and £$7\frac{1}{2}$ premium besides.

$$\text{Hence required premium} = \frac{3700}{1} \times \frac{7\frac{1}{2}}{92\frac{1}{2}} = \frac{3700}{1} \times \frac{15}{185} = £300.$$

2. Rates per cent. are frequently applied to define the profit or loss upon commercial transactions. Thus, if tea, bought at 2*s.* 4*d.* per lb., were sold at 2*s.* $7\frac{1}{2}$*d.*, the gain would be $3\frac{1}{2}$*d.* on 2*s.* 4*d.*, and therefore on £100 it would be $100 \times \frac{3\frac{1}{2}}{28} = 100 \times \frac{1}{8} =$ £$12\frac{1}{2}$, and the profit would thus be $12\frac{1}{2}$ per cent. All questions of profit and loss are examples of proportion; the following are, however, general rules:—

Rate per cent. of profit or loss = difference of cost and selling prices ÷ cost price × £100.

Selling price = cost price × £100 *plus rate of profit* ÷ £100.

£100 *plus rate of profit = selling price ÷ cost price* × £100.

In the second and third of these rules, if there is a loss, the rate of loss must be *subtracted* from £100.

If paper is bought at 18*s.* a ream (20 quires), and sold at 1*s.* 3*d.* a quire, what is the gain per cent.?

$\frac{5}{4} \times \frac{20}{1} \times \frac{1}{18} \times \frac{100}{1} = \frac{1250}{9} =$ £$138\frac{8}{9}$. Therefore gain $= 38\frac{8}{9}$ per cent.

A man buys $2\frac{1}{2}$ cwt. of tobacco, hoping to make a profit of 20 per cent. by selling it at 6*s.* per lb. He finds, however, that a fifth part of it is so much damaged as to be worth to him only 2*s.* 6*d.* per lb. Supposing that he charges 6*s.* 6*d.* per lb. for the remainder, what is his gain per cent., and what is his total gain?

He sells $\frac{4}{5}$ at 6*s.* 6*d.* and $\frac{1}{5}$ at 2*s.* 6*d.* Now $\frac{4}{5} \times \frac{13}{2} = \frac{26}{5}$ and $\frac{1}{5} \times \frac{5}{2} = \frac{1}{2}$, therefore average selling price $= \frac{26}{5} + \frac{1}{2} = \frac{57}{10}$*s.* The cost price $= \frac{100}{120} \times \frac{6}{1} = 5$*s.* And $\frac{57}{10} \times \frac{1}{5} \times \frac{100}{1} = 114$. Therefore the gain is 14 per cent., and the total gain is $\frac{280}{1} \times \frac{1}{4} \times \frac{14}{100} = \frac{49}{5} =$ £9. 16*s.*

If cheese can be sold for $7\frac{1}{2}$*d.* a lb., at a profit of 20 per cent., at what price must it be sold per lb. to gain 36 per cent.?

$$\text{Here } \frac{15}{2} \times \frac{136}{120} = \frac{17}{2} = 8\tfrac{1}{2}d.$$

3. Rates per cent. are also employed in statistics, which

are numerical reports on various subjects. Thus, if out of 11,479 criminals, 2735 could read and write well, 1649 could do both indifferently, and 3627 could read only, the first class would form a percentage of $\frac{273500}{11479}$, or 23·826 nearly of the whole number. The second and third classes would be $\frac{164900}{11479}$ and $\frac{362700}{11479}$, or about 14·365 and 31·597 per cent. of the whole respectively, and the number, not included in these classes would be $\frac{346800}{11479}$, or about 30·212 per cent.

In a factory where 560 workmen are employed, 10 per cent. receive 35*s.* a week, 25 per cent. receive 30*s.*, 50 per cent. receive 25*s.*, and the remainder 20*s.* Find how many workmen there are in each class, and the average rate of wages?

The first class will contain $560 \times \frac{1}{10}$, or 56 men; the second $560 \times \frac{1}{4}$, or 140; the third $560 \times \frac{1}{2}$, or 280. Since 85 per cent. of the men are thus accounted for, there remains in the fourth class 15 per cent., and this will amount to $560 \times \frac{3}{20}$, or 84 men. The average rate of wages will be $(\frac{1}{10} \times 35) + (\frac{1}{4} \times 30) + (\frac{1}{2} \times 25) + (\frac{3}{20} \times 20) = 3\frac{1}{2} + 7\frac{1}{2} + 12\frac{1}{2} + 3 = 26\frac{1}{2} =$ £1 6*s.* 6*d.*

Ex. 38.

1. What is the premium on a policy of insurance for £3500 at £2 8*s.* 9*d.* per cent.?

2. If an agent's commission is £232 13*s.* 10*d.* on transactions to the amount of £4653 16*s.* 8*d.*, what does he charge per cent.?

3. The premium and duty on a fire insurance are together 6*s.* 9*d.* per cent., and amount to £6 8*s.* 3*d.* Find the sum assured.

4. For what sum must goods worth £4800 be insured at 3 per cent., so that in case of loss the owner may recover both goods and premium?

5. At what price per lb. must soap, which cost £1 10*s.* per cwt., be sold so as to gain 40 per cent.?

6. If eggs are bought at 6*s.* 8*d.* per 100, and on the average only $\frac{9}{10}$ are actually sold, how much per cent. is gained by selling 5 for 6*d.*?

7. The actual cost of 6 pianos being £114, a tradesman wishes to fix such a price as to allow him to take off 5 per cent. for cash payment, and yet gain 20 per cent. on his outlay. What must he charge for each piano?

8. A draper, after selling $\frac{3}{8}$ of a quantity of merino at 20 per cent. profit, has the remainder damaged by a fire, and is obliged to sell it at 15 per cent. loss. His gain on the whole being £3 9*s.* 9*d.*, find the cost price of the merino.

9. If cheese could be sold at £3 7*s*. 1*d*. per cwt., with a profit of 15 per cent., what must be charged per lb. so as to gain 28 per cent.?

10. By selling 24 instead of 25 bundles of firewood for 1*s*., there is an additional gain of 5 per cent. on the outlay. What was the cost price of 1,000?

11. Bohemian glass contains 76 per cent. of silica, 15 of potash, 8 of lime, and 1 of alumina. Find the quantity of each respectively in 1 cwt. of glass.

12. In the 20 years from 1841 to 1861, the population of England increased from 15,914,148 to 20,066,224, that of Scotland increased from 2,620,184 to 3,062,294, and that of Ireland decreased from 8,175,124 to 5,798,233. Find approximately (to two places of decimals) the rate per cent. of increase or decrease in each of the three, and in the three taken together.

In all questions concerning concrete quantities that have hitherto been given, it has been assumed that our English standards of time, space, weight, and value are to be employed. But there are a great variety of questions of practical importance which involve not merely our standards of measurement, but those of foreign nations. As far as time is concerned there is no discrepancy, as the length of a day, and the succession of light and darkness has been fixed for us, and is not settled by ourselves; but with regard to space, weight, and value, different nations have adopted different systems of measurement. As to space and weight, however, this involves very little difficulty. When we have once ascertained that 100 English yards are equal in length to 91·43 French metres, it is merely a simple example of proportion to express any length whatever in the French measures: and similarly with respect to any other instance, it is merely necessary to know the ratio between the quantity expressed by the English word and the quantity expressed by the foreign word, and the whole relation between the two systems of measurement is determined. For example :—

If in Turkey an 'oke' of 400 drams = 2·83 lb. avoirdupois, find how many okes and drams there are in 17 cwt. 11 lbs.

Here $\frac{17 \text{ cwt. } 11 \text{ lbs.}}{2\cdot83 \text{ lbs.}} = \frac{1915}{2\cdot83} = 676\cdot678$. Therefore weight = 676 okes 271·2 drams.

100 English gallons being equal to 454·34 French litres, reduce 8763·5 litres to English measure.

Here $\frac{8763\cdot5}{454\cdot34} \times 100$ gallons $= 1928\cdot8$ gallons.

Before proceeding to examine how far a similar method of comparison can be applied to standards of value, it is necessary to give a brief account of some of the most important of these standards.

In England, gold used for coin, or *standard gold*, contains 22 parts by weight of pure gold out of 24, and is called 22 *carats* fine; gold of any other degree of purity being similarly described. Thus, that which jewellers use for making watch cases is generally 18 carats fine, that is, contains $\frac{18}{24}$ of its weight of pure gold. No charge is made for coining gold, and any amount, worth not less than £10,000, brought to the Mint, will be made into coin, and redelivered in that form, without any deduction whatever.

The weight of a sovereign being 123·274 grains troy, it follows that an ounce of standard gold is worth $\frac{480}{123\cdot274}$ sovereigns, or £3 17*s.* 10½*d.* The Bank, however, when buying gold in bars, only give £3 17*s.* 9*d.*, and when selling gold charge the full price. Standard silver contains 37 parts by weight of pure silver, out of 40. Out of a troy pound of this metal there are coined 66 shillings, so that the value of an ounce of standard silver, as deduced from this fact, would be $\frac{66}{12}$ shillings, or 5*s.* 6*d.* This is called the *mint price* of silver, but it is considerably in excess of the ordinary price of uncoined silver. The reason for this difference is, that the ratio of the values of silver and gold coins is fixed by authority, so that a shilling circulates as being worth $\frac{1}{20}$ of £1, while in reality it is probably not worth more than $\frac{1}{22}$. This arbitrary regulation is made because we choose to have all our measures of value dependent upon a gold standard, the sovereign; but it

would produce most pernicious consequences, and lead to endless confusion, if proper safeguards were not adopted.

Thus, silver being practically worth about *5s.* an ounce when uncoined, and *5s. 6d.* when coined, it would be easy for any one to take a quantity of silver to the Mint, demand to have it coined, employ the money thus obtained in buying more silver, and repeating the process with a loss to the Government every time. This consequence is prevented by vesting in the Crown solely the power of determining when and to what amount silver shall be coined; and this is so regulated as to keep the quantity such as is suited to the requirements of the nation. But another evil has to be guarded against. Silver being worth less than its circulating value, will not command that value abroad, while a sovereign will never be subjected anywhere to much depreciation. Consequently, most persons would think the gold decidedly preferable, and if they had money to pay, would choose to pay in silver, while they would try to bargain that debts to them should be paid in gold. The natural effect of this would be that the ratio between the values of gold and silver coins would alter in spite of the Government. To prevent this and other pernicious hindrances to trade, it is enacted that silver is not a legal tender for more than *40s.*

In France the standard of value is the *franc*, a silver coin containing 9 parts by weight of pure silver, out of 10, and of such a weight that 200 are coined from a kilogramme, a weight containing 15434 grains troy. The *napoleon* is a gold coin, containing 9 parts by weight of pure gold, out of 10, and of such a weight that 155 are coined from a kilogramme.

In Amsterdam the standard of value is the florin, containing about $147\frac{1}{30}$ grains of pure silver; and in Hamburgh the standard is the mark, containing about 106·1 grains. In these two countries, especially, it is however common to reckon amounts in bar gold or silver, the Amsterdam mint receiving gold at a fixed rate, and silver at a variable rate,

and the Hamburgh bank receiving silver at a fixed, and gold at a variable rate.

The above particulars with regard to foreign money have been chiefly obtained from 'Tate's Modern Cambist,' and 'Tate's Counting-house Guide,' and from these works the pupil may obtain full information on all matters connected with this subject. They have been given in order to furnish materials for showing how far the former methods of comparison may be applied to standards of value.

Suppose that it were required to compare the values of a French franc and an Amsterdam florin. The franc contains $\frac{15434}{200}\times\frac{9}{10}=69\cdot453$ grains of pure silver, and therefore the florin$=\frac{147\cdot0\dot{3}}{69\cdot453}=2\cdot117$ francs. Assuming that the coins are actually as good as they are represented to be, this would be the *par of exchange* between Paris and Amsterdam.

Next, let us try to make a comparison between the French franc and the English shilling. Here we are met at once by the difficulty that the English shilling has an artificial value, so that it is impossible to derive a correct conclusion from merely comparing the quantity of fine metal in each. A shilling is $\frac{1}{20}$ of a sovereign, which is our standard of value, and the franc and the sovereign cannot be compared unless we commence by assuming some ratio to exist between the values of gold and silver. This is done somewhat roughly by considering standard silver as worth 5*s.* an ounce. Thus, 1 franc contains 69·453 grains of pure silver, and £1 buys 4 ounces of silver $\frac{37}{40}$ fine. Hence £1$=\frac{4}{1}\times\frac{37}{40}\times\frac{480}{1}\times\frac{1}{69\cdot453}=25\cdot5725$ francs. On this or any other estimate of the value of silver, tables of pars of exchange may be calculated. No estimate, however, of the comparative worth of silver and gold can be perfectly correct, because the values of these metals are subject to continual variation, and the ratio between these values varies also. Consequently, the par of exchange between two countries, one of which uses a gold and the other a silver standard,

cannot be determined so as always to be accurately true. At the same time, by making an approximate estimate of the worth of one metal, expressed in terms of the other, we may obtain a par that will be practically useful as a basis for calculation. The assumption just made was that one sovereign was worth four ounces of standard silver. Another assumption is also made for the purpose of comparison, namely, that a French napoleon is worth 20 francs. This is its nominal value in circulation, and is generally perhaps a trifle below its actual value. Now, a napoleon contains $\frac{15434}{155} \times \frac{9}{10}$ grains of pure gold, and a sovereign contains $\frac{123 \cdot 274}{1} \times \frac{22}{24}$. Therefore £1 is worth $\frac{123 \cdot 274}{1} \times \frac{22}{24} \times \frac{10}{9} \times \frac{155}{15434} = 1 \cdot 26094$ napoleons. Consequently, on the assumption just mentioned, we should find that £1 was worth $20 \times 1 \cdot 26094 = 25 \cdot 2188$ francs. Of the two pars of exchange between London and Paris, that first found, of 25·5725, is called the *silver par*, and the latter, of 25·2188, is called the *gold par*.

The above is a sketch of the principles upon which comparisons of the values of different kinds of money may be made, and of their application to the particular case of England and France. In the works already referred to, the currencies of all countries of commercial importance are described, and their relative values estimated. This explanation serves as an introduction to the very difficult subject of Exchange, of which nothing more than the slightest outline can be here attempted.

Suppose a London merchant buys a quantity of French goods. One obvious mode of paying for them would be to send over to France a sufficient quantity of gold or silver to satisfy the debt. The proper amount to be thus sent might be estimated according to the principles already laid down. If silver be used, the silver sent and the number of francs in the price should contain equal quantities of the pure metal. If gold be used, the gold sent should be enough to cover the price when estimated in gold, which, as has

been stated, is in France valued at a perpetually varying amount. By taking any of these modes of payment, however, the merchant incurs the expense of freight and insurance, and the outlay for these purposes is utterly unproductive expenditure. In reality, there are in all parts of the world merchants who are constantly requiring to remit coin or *bullion*, as uncoined gold or silver is called, to other countries. It is evident, therefore, that if large sums have to be sent from England to France by some persons, and large sums have to be sent from France to England by others, it would only require mutual understanding and arrangement among the senders and receivers to enable them to balance their accounts, so that merely the difference between the total sum due and the total sum receivable might be actually carried across sea. This mutual understanding is effected by means of *Bills of Exchange*, which, though not money, are papers representing money, and may be sent at the mere cost of postage. A London merchant would, therefore, send to France, in exchange for goods, a bill authorising the payment of a sum of money in London, and his correspondent would find some one in Paris who wanted to buy goods in London, and sell him the bill for remittance to England. If the values of the goods bought happened to be equal, the whole business might be effected with no other circulating medium than ink and paper. Whether we consider the case of two merchants only, or whether we examine the whole of the vast system of commercial exchange, the principle is the same. *Bills of exchange are employed to save the expense and risk of remitting coin and bullion, so that only the final balances of accounts between countries need be so settled.* They consequently perform the same office with regard to different nations, that cheques on bankers do for individuals. The business of mediating between those who want bills and those who hold them is managed by brokers, who are ready either to buy or sell. Sometimes bills on a particular country

are much wanted, as large payments have to be made there, and the price of such bills would rise; for a merchant is naturally ready to pay a somewhat increased sum rather than incur the loss of having to pay for transmission of gold. At other times the bills may be plentiful, with comparatively few demands for them. From these causes arise variations in the *course of exchange*, which, as far as London is concerned, means the price paid there for bills on other countries. Thus, on Nov. 20th, 1863, Paris bills were to be bought at the rate of from 25·275 to 25·35 francs per £1, Amsterdam bills at from 11·85 to 11·9 florins per £1, and similarly for a long list of other countries. In most cases, the bills are supposed not to be due immediately, but at three months' date. Here it is merely requisite to deduct the customary rate of 4 per cent. per annum, and the *short exchange*, as it is termed, will be found approximately. Thus, Hamburg prices at three months being from 13 marks 8½ schillings to 13 marks 9 schillings per £1 (1 mark = 16 schillings), the short exchange would be from 13*s.* 6¼*d.* to 13*s.* 6¾*d.*

It must be remembered that this variation in ratio of exchange is not without limit. One obvious limit is the cost attending the remittance of gold, for if the price of bills was above par by an amount greater than this cost, gold would be sent instead of bills. But other causes begin to operate before this limit is reached. Suppose that England has to make large payments to France, and that the price of bills is consequently very high. Merchants, whose interest it is to avoid as long as possible the expense of remitting gold, might send papers which were worth gold, such as government securities, stocks, shares, &c. These would answer somewhat roughly the purpose of bills of exchange. Again, the price of bills on France being high, the payments to be made by other countries to England might be, as it were, made to flow through France. As an exaggerated instance of such a proceeding, suppose that, in London, bills

on Paris are at 1 per cent. premium, and that in Paris bills on Amsterdam are at ½ per cent. premium. Then an Amsterdam merchant, who had to remit to England, might send a bill, say for £100, to Paris, have it there sold for a Paris bill for £100 10*s*., and have that second bill sent to London, where it would be worth £101 10*s*. The Amsterdam merchant would, therefore, find it his interest to assist in arranging the commercial transactions of England and France. It will be seen, from this instance, how any unusual variation in the course of exchange has a tendency to correct itself. The rate of exchange practically established by means of sending bills circuitously through one or more places, is called an *arbitrated rate of exchange*, and the maxim with respect to such a mode of payment is—If the difference between the direct and arbitrated rates is sufficient to cover the increased cost of brokerage, agency, or other expenses, indirect remittances are best; if not, then direct remittances are best.

It has been attempted in a few pages to give some account of a very complicated subject—complicated, perhaps, not so much by absolute difficulties of principles as by the number of technical terms and details of commercial arrangement by which it is surrounded. Problems in exchanges do, indeed, like almost all arithmetical questions, resolve themselves into instances of proportion, but the difficulty is properly to take account of all circumstances, and at the same time clearly see the main question through all the entanglements of the commercial terms and commercial customs of perhaps several different countries. It is hoped, however, that the first elementary principles may be understood from the foregoing explanation.

A good illustration of some of the principles that have been explained is contained in the following quotation from the 'Times' of November 25th, 1863:—

'The quotation of gold at Paris is about 2½ per mille premium, and the short exchange on London is 25·30 per £1 sterling. On comparing these rates with the English Mint price of £3 17*s*. 10½*d*. per ounce for

standard gold, it appears that gold is nearly $\frac{3}{10}$ per cent. dearer in London than in Paris.'

'By advices from Hamburg the price of gold is $425\frac{1}{2}$ per mark, and the short exchange on London is 13·5 per £1 sterling. Standard gold at the English Mint price is therefore about $\frac{1}{10}$ per cent. dearer in Hamburg than in London.'

It has been explained that the gold par of exchange with France is founded on the fact that coin to the value of 3100 francs can be made from 1 kilogramme (15434 troy grains) of gold of the French standard, that is, $\frac{9}{10}$ fine. In the comparison of the 'Times,' however, the charges of the Paris Mint, amounting to 6 francs per kilogramme, are taken into consideration. Hence, remembering that English gold is $\frac{11}{12}$ fine, and valuing it at the Mint price, we should have as the worth of £1 in francs:

$$\frac{480}{15434} \times \frac{160}{623} \times \frac{10}{9} \times \frac{11}{12} \times \frac{3094}{1} = 25{\cdot}17 \text{ nearly.}$$

It should be noticed that this result is entirely independent of any variation in rate of exchange or premium on gold in Paris, and consequently the result 25·17 is called the *fixed number* for bullion operations with Paris, corresponding to the English Mint price. As gold in Paris is at $2\frac{1}{2}$ per mille premium, the worth of £1 would be $25{\cdot}17 \times 1{\cdot}0025$, or 25·233 nearly. And since the rate of exchange makes £1 worth 25·30 in London, gold is dearer in London by about ·067 in 25·30, or between $\frac{2}{10}$ and $\frac{3}{10}$ per cent.

The mark of gold mentioned in the Hamburg advices is the 'Cologne mark,' a weight equivalent to 3608 grains troy, and this is sold for 425 marks 8 schillings (16 schillings = 1 mark). Hence, from these facts, we find the price of £1 in Hamburg to be $\frac{160}{623} \times \frac{480}{3608} \times \frac{11}{12} \times 425\frac{1}{2}$. The first three of these four quantities do not depend upon any rate of exchange, and may therefore be multiplied to give a *fixed number* for bullion operations with Hamburg, $\frac{160}{623} \times \frac{480}{3608} \times \frac{11}{12} = {\cdot}03132$. Now, ${\cdot}03132 \times 425{\cdot}5 = 13{\cdot}327$ marks nearly, and this is the price of £1 in Hamburg. In London £1 is worth 13·3125, and hence gold is dearer in Hamburg by ·014 in 13·327, or about $\frac{1}{10}$ per cent.

Ex. 39.

1. The Roman 'canna,' containing 8 'palmi,' is equal to 78·35 English inches. Find how many canne and palmi there are in 50 yd. 2 ft. 7 in.

2. If 100 Portuguese pounds are equal to 101·18 lb. avoirdupois,

and the French kilogramme is equal to 15434 grains troy, how many Portuguese pounds are there in 141·652 kilogrammes?

3. The 'hectare' is a French measure of surface, being a square, each side of which is 100 metres, or 109·36 yards. Find approximately how many hectares are equivalent to 10 English acres.

4. Reduce £187 13*s.* 4*d.* into francs and centimes (1 franc = 100 centimes), the exchange being £1 = 25·20 francs.

5. In Madrid, 85 dollars of plate (1 dollar = 8 reals) are worth 64 hard dollars. The exchange between Madrid and Berlin makes the dollar of plate worth 1 Prussian dollar $\dot{2}\frac{8}{17}$ silver groschen (30 s.g. = 1 P. D.), and that between London and Berlin makes £1 worth 6 P. D. 27 s.g. Find the value of the hard dollar in English pence.

6. What exchange in pence per 'milrei' (1 milrei = 1000 reis) is established between London and Lisbon by bills on Paris being bought in London at 25·28 francs per £, and sold in Lisbon at 180 reis per franc?

7. Taking the fixed numbers for Paris and Hamburg at 25·17 and ·03132 respectively, find by how much per cent. gold is cheaper or dearer in those places than in London, when it is quoted in Paris at 2 per mille premium, and in Hamburg at $426\frac{1}{2}$ per mark, the short exchanges being 25·25 francs, and 13 marks $4\frac{5}{8}$ schillings.

8. £1000 is remitted from London to Amsterdam through Paris, by means of bills bought in London at 25·40, and sold in Amsterdam at $57\frac{1}{2}$ florins per 120 francs. The direct rate between London and Amsterdam being 11 florins $19\frac{1}{2}$ stivers (20 stivers = 1 florin), find the profit from sending through Paris, $\frac{1}{5}$ per cent. being deducted for the extra brokerage.

CHAPTER VII.

MISCELLANEOUS RULES AND LOGARITHMS.

THERE still remain a few arithmetical processes which have not been explained. These will be given in the first part of this chapter, and the remainder will be devoted to some account of the nature and use of logarithms.

It is often required to divide an abstract or a concrete number into parts which shall have a given ratio to one another. For example, divide 176 into 4 parts, which shall bear to one another the same ratios as the numbers 2, 3, 5, 6. Now, if these numbers be multiplied successively by any numbers, as, for instance, 2, 5, $3\frac{1}{4}$, 7, we shall obtain sets of numbers, 4, 6, 10, 12; 10, 15, 25, 30; $6\frac{1}{2}$, $9\frac{3}{4}$, $16\frac{1}{4}$, $19\frac{1}{2}$; 14, 21, 35, 42; which manifestly bear to one another the same ratios as 2, 3, 5, 6. It remains, then, to choose, out of all possible sets of this kind, the particular one where the sum of the terms equals 176; and since the sum of 2, 3, 5, 6, is 16, it is clear that their sum, and consequently the separate numbers themselves, must be multiplied by 11 to form the set required, which will therefore be 22, 33, 55, 66. From this and similar instances, we may deduce the general rule for dividing a quantity into parts having given ratios. *Divide the quantity by the sum of the numbers expressing the ratios, and multiply the quotient by each of the numbers in succession.* The rules which are, in many Arithmetics, called 'Simple Fellowship' and 'Compound Fellowship,' are merely instances of this. In the first, the profits of any trading firm are to be divided among the partners proportionally to the amounts invested by them in the business. In the second, the profits are to be divided in proportion both to the amounts, and to the times during

which they have been invested, and each share must therefore be proportional to the product of the numbers representing the money and the time in the case of each partner.

Divide £356 6*s*. 2*d*. in the proportion of the numbers $2\frac{1}{3}$, $3\frac{1}{4}$, $4\frac{1}{5}$, $5\frac{1}{6}$.

$$2\tfrac{1}{3}+3\tfrac{1}{4}+4\tfrac{1}{5}+5\tfrac{1}{6}=14+\frac{20+15+12+10}{60}=14\tfrac{57}{60}=14\tfrac{19}{20}=\frac{299}{20}.$$

And $356\frac{37}{120}\div\frac{299}{20}=\frac{42757}{120}\times\frac{20}{299}=\frac{143}{6}$. Therefore the shares are $\frac{143}{6}\times\frac{7}{3}=\frac{1001}{18}=$ £55 12*s*. $2\frac{2}{3}$*d*.; $\frac{143}{6}\times\frac{13}{4}=\frac{1859}{24}=$ £77 9*s*. 2*d*.; $\frac{143}{6}\times\frac{21}{5}=\frac{1001}{10}=$ £100 2*s*.; and $\frac{143}{6}\times\frac{31}{6}=\frac{4433}{36}=$ £123 2*s*. $9\frac{1}{3}$*d*.

A, B, C, are partners, and each puts £1200 into the business, but B draws out £75, and C draws out £50 at the end of 3, 6, and 9 months. At the end of 12 months the profits, amounting to £864 10*s*., are to be divided. What should each receive?

Here A has £1200 in for 12 months. B has £1200 for 3, £1125 for 3, £1050 for 3, and £975 for 3, which is the same as £4350 for 3 months. C has £1200 for 3, £1150 for 3, £1100 for 3, and £1050 for 3, altogether £4500 for 3 months. Therefore the shares must be proportional to the numbers 14400, 13050, and 13500. These numbers are as 32 : 29 : 30, and the sum of these latter = 91. Hence A's share = $864\frac{1}{2}\times\frac{32}{91}=\frac{1729}{2}\times\frac{32}{91}=$ £304; B's share $=\frac{1729}{2}\times\frac{29}{91}=\frac{551}{2}=$ £275 10*s*.; and C's share $=\frac{1729}{2}\times\frac{30}{91}=$ £285.

Divide £963 1*s*. among 5 persons, giving the second half as much again as the first, and each of the others half the sum of the two preceding shares.

If the 1st share be represented by 1, the 2nd must be $\frac{3}{2}$, the 3rd one-half of $1+\frac{3}{2}$, or $\frac{5}{4}$, the 4th one-half of $\frac{3}{2}+\frac{5}{4}$, or $\frac{11}{8}$, and the 5th one-half of $\frac{11}{8}+\frac{5}{4}$, or $\frac{21}{16}$. To get rid of fractions, let these proportional quantities be multiplied by 16, then the shares must be as 16 : 24 : 20 : 22 : 21. The sum of these numbers is 103. Hence 1st share $=963\frac{1}{20}\times\frac{16}{103}=\frac{19261}{20}\times\frac{16}{103}=\frac{187}{20}\times\frac{16}{1}=\frac{748}{5}=$ £149 12*s*.; 2nd share $=\frac{187}{20}\times\frac{24}{1}=\frac{1122}{5}$ = £224 8*s*.; 3rd share $=\frac{187}{20}\times\frac{20}{1}=$ £187; 4th share $=\frac{187}{20}\times\frac{22}{1}=\frac{2057}{10}$ = £205 14*s*.; 5th share $=\frac{187}{20}\times\frac{21}{1}=\frac{3927}{20}=$ £196 7*s*.

Ex. 40.

1. Divide the number 2191 into parts which shall have to one another the same ratios as the numbers 4, 3, $7\frac{1}{4}$, $11\frac{5}{8}$.

2. Three partners invest in a business £850, £570, and £400 respectively. After trading for a few years, they divide their capital, then amounting to £3294 4*s.* How much should each partner receive?

3. Three towns, containing 1750, 3200, and 2640 inhabitants respectively, are taxed in proportion to their population. The total amount paid being £2846 5*s.*, what did each town contribute?

4. If 6 oz. of gold, 18 carats fine, are mixed with 4 oz. 12 carats fine, how much pure gold will there be in each ounce of the mixture?

5. Divide the fraction $\frac{13}{19}$ into three parts, which are proportional to the numbers 7, 9, and 11.

6. In dividing the profits of a firm, it is found that A has had £2000 in the business for 12 months; B has had £1500 in for 8 months, and then increased it to £2500; and C has had £2400 in for 3 months, and then drawn out £1500. If B's share is £159 2*s.* 6*d.* more than C's, how much ought A to receive?

7. Divide £665 among 4 persons, so that the share of the first may be to that of the second, as 7 : 9; the share of the second to that of the third, as 11 : 13; and the share of the third to that of the fourth, as 15 : 17.

8. In winding up the affairs of a company which is insolvent, it is found that the good, bad, and doubtful debts owing to the company are in the proportion of the numbers 9, 5, and 3. The bad debts are estimated as worth nothing, and the doubtful as worth half their nominal value. Afterwards it turns out that $\frac{1}{25}$ of the bad debts are recovered. In what proportion will the dividend received by the creditors be increased from this cause?

The number which, when multiplied by itself, produces a given number, is called the *square root* of that number. Thus 3 is the square root of 9, 11 is the square root of 121, and 16 of 256. The number which, when multiplied by itself twice, produces a given number, is called the *cube root* of that number. Thus, 2 is the cube root of 8, 3 is the cube root of 27, and 5 of 125. When the continued product of a number, repeated four, five, six, or more times, amounts to a given number, the first is called the *fourth, fifth, sixth, or lower root* of the second. Thus 2 is the 5th

root of 32, 3 is the 6th root of 729, and 11 is the 4th root of 14641. Consequently, the terms square root and square, cube root and cube, fourth root and fourth power, seventh root and seventh power, &c., are respectively correlative to one another; as, for example, 6 is the *square root* of 36, 36 is the *square* of 6; 7 is the *cube root* of 343, 343 is the *cube* of 7; 5 is the *fourth root* of 625, 625 is the *fourth power* of 5; 3 is the *seventh root* of 2187, 2187 is the *seventh power* of 3.

The problem of finding the square root of any given number, is one of constant occurrence in practice, and the process by which it is accomplished depends upon the following principle:—*The square of the sum of any two numbers is equal to the sum of their squares, together with twice their product.* Thus $39=25+14$, and the square of 39 is 1521, and 1521 is equal to $625+196+700$, that is, to the squares of 25 and 14, and twice the product of 25 and 14. Let another instance be taken, where the first number is considerably larger than the second, as when 39 is separated into 37 and 2. Then, taking the squares of both, and twice their product, we have $1369+4+148=1521$, the square of 39 as before; but here, and in similar instances, it may be noticed, that the largest term is the first square, and that the second square, which is the smallest term, is considerably smaller than the term representing twice the product. Suppose we attempt, by the aid of the principle just stated, to find the square root of the number 68,211081. Now the square root of 64,000000 is 8000, and of 81,000000 is 9000. Hence the required square root must be between 8000 and 9000, or 8000 and some unknown number besides. Therefore 68,211081 must be equal to the squares of 8000 and the unknown quantity, and 16000 times the unknown quantity besides. The square of 8000 is 64,000000, and consequently, if we leave out of sight, for a moment, the square of the unknown quantity, which is small compared with the

ther terms, we find that 4,211081 must be somewhere bout 16000 times that quantity, and hence it is about $\frac{,211081}{16000}$ or between 200 and 300. Having thus arrived at ne step nearer the real root, we suppose it made up of 200 and something more. The square of 8200 is equal to he squares of 8000 and 200, together with 16000×200, or o $64,000000 + (16200 \times 200)$. The former has been already ubtracted from the original number, leaving 4,211081, nd now, subtracting the latter, 3,240000, there remains 71081, for which we have still to account. Pursuing the rocess, we consider this number as about 2×8200, or 6400 times the remainder of the root. Now, $\frac{971081}{16400}$ is etween 50 and 60, and we therefore consider the root as 250, and something more. The square of 8250 is greater ian the square of 8200 by 16450×50, or 822500, and e have now left $971081 - 822500$, or 148581. As before, $\frac{48581}{\times 8250}$ or $\frac{148581}{16500}$ is a little more than 9. And the differnce between the squares of 8259 and 8250 is 16509×9, r 148581, and thus the whole of the original number has een accounted for. The work that has just been gone ırough, may be shown in the following form;—

	68211081	(8000 + 200 + 50 + 9
square of 8000 =	64000000	
	4211081	which ÷ 16000 gives more than 200.
16200 × 200 =	3240000	
	971081	which ÷ 16400 gives more than 50.
16450 × 50 =	822500	
	148581	which ÷ 16500 gives more than 9.
16509 × 9 =	148581	

This form will evidently be much shortened, if we only etain such portions as are absolutely necessary to carrying n the work. It will then appear as follows:—

The dots are placed over every alternate figure, because it is necessary to bring down two figures at each operation to carry on the subtraction, and there must always be in the quotient a figure for each period of two figures brought down. In the instance given, the number is a complete square, and thus its root may be found exactly, but where the number is not a complete square, we may approximate to the root as far as we please. Since decimals follow the same law as whole numbers, the square root of a decimal is found in precisely the same manner as that of a whole number. Care must, however, be taken not to include the place of units and the place of tenths in the same period. Thus 241·375 must be pointed $\dot{2}4\dot{1}\cdot3\dot{7}5\dot{0}$, and not $2\dot{4}1\cdot\dot{3}7\dot{5}$. All that is necessary to prevent a mistake on this point, is to see that there is a dot over the place of units. The place of the decimal point in the root is determined by the simple consideration, that integral periods give integral figures in the root, and that decimal periods give decimal figures.

```
         68211081(8259
         64
  162)    421
          324
 1645)     9710
           8225
16459)     148581
           148581
```

The following is the formal statement of the rule for extracting the square root :—*Divide the given quantity into periods, by placing a dot over the unit figure, and over every alternate figure towards the left, and if there are decimals, towards the right also. Place, as the first figure of the root, that which is next less than the square root of the first period. Square this first figure, subtract it from the first period, and to the remainder add on the second period, calling the number thus composed the* minuend. *Form a* trial divisor *by doubling the first figure of the root, and see how often it is contained in the minuend, omitting its last figure. Place the result both as second figure of the root, and at the end of the trial divisor. The latter will then become an actual divisor, and must be multiplied by the figure just found, and the product subtracted from the minuend. The*

remainder, with the next period added, will form a new minuend, the new trial divisor will be the double of the figures in the root, and the process may be continued as far as necessary.

It will often happen that the figure found by dividing by the trial divisor will, when multiplied by the actual divisor, produce a number greater than the minuend from which it has to be subtracted. In such a case, the next less figure must be substituted, or, if that again be too great, the next less figure, and so on. This uncertainty, for the moment, about the true figure, is much more likely to occur at the beginning than at any after period of the process. To exemplify it, we will find the square roots of 232324, 28561, and 1413721:

```
      232324(482          28561(169          1413721(1189
      16                  1                  1
      ----                ---                ----
 88)  723            26)  185           21)    41
      704                 156                  21
      ----                ---                  ----
962)  1924          329)  2961         228)    2037
      1924                2961                 1824
      ----                ----                 ----
                                       2369)   21321
                                               21321
                                               -----
```

In the first of these examples, the trial divisor 8 would be contained 9 times in 72, but if we put 9 in the root, we should have to multiply 89 by 9, making 801, which would be too large to take from 723. We therefore put the next less figure, 8, in the root. In the second example, the trial divisor 2 is contained 9 times in 18, but 9 times 29 is 261, which is too large; 8 times 28=224; 7 times 27=189; and as each of these is too large, we must try the figure 6. In the third example, the first trial divisor would give 2, but $21 \times 2 = 42$, is too large, so we use the figure 1 instead. Again, the second trial divisor, 22, would give 9, since 9×22, or 198, is less than 203. But if 9 were tried, the product 9×229, or 2061, would be found too large, and so 8 is the third figure of the root.

The following are examples of finding approximately the square roots of quantities which are not exact squares:—

Find the square roots of 29, 144485·2764, and ·00000006.

```
      29(5·385, &c.                 144485·2764(380·11, &c.
      25                             9
     ----                           ---
 103) 400                     68)   544
      309                           544
     ----                           ----
1068) 9100                  7601)    8527
      8544                           7601
     -----                          -----
10765) 55600               76021)    92664
       53825                         76021
       -----                         -----
        1775                         16643

              ·00000006(·0002449, &c.
                      4
                     ---
               44)   200
                     176
                     ----
              484)   2400
                     1936
                     -----
             4889)  46400
                    44001
                    -----
                     2399
```

It may be shown by algebra, that additional figures in the root may be found by dividing the remainder by the trial divisor, but that only one less than the number already found can be thus obtained. In the first of the preceding examples, therefore, three figures might be found by dividing 1775 by 10765, in the second four figures by dividing 16643 by 76021, and in the third three figures by dividing 2399 by 4889. Approximating in this way, the roots would be, 5·385165, 380·112189, and ·000244949.

Ex. 41.

Find the square roots of—

1. 34562641.
2. 76825225.
3. 153·685609.
4. 45152·0001.
5. 119.
6. 73·952.
7. Find the length of one side of a square field which contains 2 acres 3 roods 555 yds. 0 ft. 81 in.
8. Each man of a party at a tavern pays, as his share of the reckoning, three times as many farthings as there are men in the party. The total reckoning being £3 12*s.* 3*d.*, how many men are there?

The process for finding the cube root of a quantity depends upon principles which cannot well be understood by those who are unacquainted with algebra. No attempt will therefore be made at explanation, and the process itself is all that will be stated. Of the various forms in which the work is arranged in different books on arithmetic, that in Colenso's Arithmetic is the best, and, as it possesses some peculiarities, could not fairly be given here without acknowledgement.

Divide the given quantity into periods, by placing a dot over the unit figure, and over every third figure to the left, and, if there are decimals, to the right also. Place, as the first figure of the root, that which is next less than the cube root of the first period. Cube this first figure, subtract it from the first period, and to the remainder add on the second period, thus forming the first minuend. In a column some distance to the left, place three times the part of the root found at each stage of the process. The first trial divisor is placed at the head of a column a little more to the right, and is the triple of the square of the first figure in the root, and two ciphers annexed. Another figure of the root being found by this trial divisor, place it in the root, and also at the end of the number in the left-hand column, and when thus increased, multiply the latter by the figure in the root. Place the product under the trial divisor, and add the two together: this will give the first actual divisor, from which we obtain a second remainder, and, by bringing down another period, a second minuend. The second trial divisor is found by adding together the actual divisor, the product just mentioned, and (which need not be actually put down) the square of the last found figure of the root, two ciphers being annexed to the sum. This process can be continued as far as necessary.

As in square root, it sometimes happens that the figure obtained by the trial divisor is too large, and the next less has to be tried; and as in square root, additional figures in the root may be found by ordinary

division by the last actual divisor. The number of figures thus obtained must, however, not exceed two less than the number already found.

Find the cube root of 14760213677, and of 647, to three places of decimals.

		14760213677 (2453			647 (8·649
		8			512
64	1200	6760	246	19200	135000
	256			1476	
	1456	5824		20676	124056
725	172800	936213	2584	2218800	10944000
	3625			10336	
	176425	882125		2229136	8916544
7353	18007500	54088677	25929	223948800	2027456000
	22059			233361	
	18029559	54088677		224182161	2017639449
					9816551

By dividing by the last divisor, two more figures, 04, may be obtained, thus making the cube root of 647 equal to 8·64904. (Note L.)

Ex. 42.

Find the cube roots of—

1. 44361864. 2. 178453547. 3. 115074924·544.
4. 5913. 5. 11. 6. 85·96.
7. What is the length of the edge of a cubical water tank that will just contain 700 gallons?

Questions in arithmetic are often proposed which depend upon the *relative velocity* of two bodies in motion, either in the same or in opposite directions, that is, the velocity with which they are approaching to or receding from one another. Strictly speaking, we know nothing practically of absolute velocity. When we talk of a man walking at the rate of four miles an hour, we mean that his relative velocity, with regard to some place on the surface of the earth, is four miles an hour, for, in reality, although we do not notice it, he is moving round the earth's centre at the rate of between six and seven hundred miles per hour, and, at the same time, round the sun at the rate of about six hundred and seventy thousand, while he may be, and probably is, carried

through space in company with the whole solar system, at other and as yet uncalculated velocities. To enable us to determine the absolute motion of anything, we ought to compare it with some point which is at rest, and throughout the whole universe we do not know of any such point. In the ordinary business of life, however, we consider the earth at rest, and that all velocities relative to points on its surface may be taken as absolute velocities. Of two moving bodies, each will, in general, have a motion relative to the other; that is to say, will change either its distance from, or its position with respect to, the other. There are, however, only two cases where the rate of this relative motion can be determined by arithmetic, namely, where the bodies are moving in the same or in opposite directions. In the former, the relative velocity will be the difference, and in the latter, the sum of their respective velocities. Thus, if a man be walking at the rate of four miles an hour, and another be at some distance behind him, riding at the rate of seven, the relative velocity of the two men is three miles an hour, and that is the rate at which the rider is overtaking the other. If two railway trains are coming in opposite directions, at twenty and twenty-five miles an hour, they are approaching each other at the rate of forty-five miles an hour, and will pass each other at that rate. The following are examples of the application of the above principles, and are of the kind usually given in examinations, or to be found in collections of arithmetical examples :—

A starts for a walk at the rate of 3 miles an hour. Half an hour afterwards B starts from the same place, and walks at the rate of $3\frac{3}{4}$ miles an hour. In what time will he overtake A?

In half an hour A will have walked $1\frac{1}{2}$ miles, and B, approaching him with the relative velocity of $\frac{3}{4}$ mile an hour, must come up with him in $1\frac{1}{2} \div \frac{3}{4} = 2$ hours.

At what time between four and five will the minute hand be exactly opposite the hour hand?

The minute hand moves through 60, and the hour hand through 5 of the minute divisions of the dial in an hour. Hence the relative velocity is 55 divisions per hour, or $\frac{11}{12}$ per minute. At 4 o'clock the minute hand is 20 divisions behind, and when the hands are opposite, it is 30 divisions before the hour hand. The number of minutes past 4 is consequently $50 \div \frac{11}{12}$, or $54\frac{6}{11}$.

On a canal a barge 50 ft. long, going at the rate of $2\frac{1}{2}$ miles an hour, meets another 60 ft. long coming in the opposite direction at $3\frac{1}{2}$ miles an hour. How long will they take to pass one another?

They will begin to pass one another when the bows of the barges are opposite to one another, and will have finished passing when the sterns are opposite. If one of the barges be supposed at rest, the other must move through a distance equal to the sum of their lengths during this interval. Hence the time of passing is the sum of their lengths divided by their relative velocity, and will therefore be $\frac{110}{5280} \times \frac{1}{6} \times \frac{60}{1} = \frac{5}{24}$ minute.

In a mile race between two boats, at the end of the first minute one is 16 feet ahead of the other, and completes the course in $6\frac{1}{3}$ minutes more. How long does the second boat take to go over the distance, and how far is it behind when the first boat comes in?

Here the velocity of the first boat is $\frac{60}{1} \div 7\frac{1}{3} = \frac{60}{1} \times \frac{3}{22} = \frac{90}{11} = 8\frac{2}{11}$ miles an hour, and the relative velocity $= \frac{16}{5280} \times \frac{60}{1} = \frac{2}{11}$ mile. Therefore the velocity of the second boat is 8 miles an hour, and the time taken to go one mile must be $7\frac{1}{2}$ minutes. Also, since in one minute the first boat gains 16 feet, in $7\frac{1}{3}$ minutes it will gain $\frac{16}{1} \times \frac{22}{3}$, or $117\frac{1}{3}$ feet.

The three following questions depend upon the same principle, and the method of treating them is best explained by working them out. Examples similar to these, and to those involving relative velocities, are given in the miscellaneous collection at the end of the book.

If A can do a certain piece of work in 8 days, and B can do it in 5 days, how long would A and B take to do it together?

A can do $\frac{1}{8}$ of the work in a day, and B can do $\frac{1}{5}$, therefore they can together do $\frac{1}{8} + \frac{1}{5}$, or $\frac{13}{40}$ in a day. Hence they can do the whole in $\frac{40}{13}$, or $3\frac{1}{13}$ days.

Two men are pumping water into a cistern. After continuing 10 minutes, and the cistern being $\frac{1}{4}$ full, one of them goes away for 5 minutes, leaving the other at work. When he returns they both pump for 7 minutes, and then find the cistern half full. The other one now goes away for 5 minutes, and then returns, and they work

together as before. What time is altogether occupied in filling the cistern?

They work for 10 minutes and fill $\frac{1}{4}$ of the cistern. Hence they together fill $\frac{1}{40}$ in a minute. Consequently, in the 7 minutes they were both working, they must have filled $\frac{7}{40}$. Hence $\frac{1}{2}-\frac{1}{4}-\frac{7}{40}$, or $\frac{3}{40}$, must have been filled by the one man in 5 minutes, and he therefore fills $\frac{3}{200}$ in a minute, and his companion fills $\frac{1}{40}-\frac{3}{200}$, or $\frac{1}{100}$. The latter works for 5 minutes by himself in filling the second half of the cistern, completing in that time $\frac{5}{100}$ or $\frac{1}{20}$, and thus leaving $\frac{1}{2}-\frac{1}{20}$, or $\frac{9}{20}$ for them to do together, which they would accomplish in $\frac{9}{20}\div\frac{1}{40}$, or 18 minutes. The whole time occupied is therefore $10+5+7+5+18$, or 45 minutes.

A, B, and C together can do a piece of work in 24 days, while A and C by themselves would take 36. If, when A works alone for 10 days, and B and C then come and help him, it is altogether finished in 30 days, find how long each would require to do it by himself.

A, B, and C together do $\frac{1}{24}$ in a day, and A and C do $\frac{1}{36}$. Hence B does $\frac{1}{24}-\frac{1}{36}=\frac{1}{72}$ in a day. Now, in the supposed arrangement, A, B, C all work together for 20 days, thus doing $\frac{20}{24}$, or $\frac{5}{6}$, and the remaining $\frac{1}{6}$ is done by A in 10 days. Therefore A does $\frac{1}{60}$ in a day, and C does $\frac{1}{36}-\frac{1}{60}=\frac{1}{90}$, and consequently C would finish the work in 90 days, B in 72, and A in 60.

A series of numbers, such that each is always greater or always less than the one before it, by a certain fixed quantity, is called an *Arithmetical Series,* and the numbers themselves are said to be in *Arithmetical Progression.* Thus, 2, 6, 10, 14, &c., is such a series, the *common difference,* or the quantity by which each term exceeds the preceding, being 4. Again, 100, 93, 86, 79, &c., is another such series, the terms continually diminishing by the common difference 7. When the first term and the common difference are known, any term of the series may be easily found. For example, in the series 2, 6, 10, 14, &c., the second term is 4 more than the first, the third term is 8, or 2×4 more, the fourth is 12, or 3×4, and so on, *any term being greater than the first by the product of the common difference and one less than the number of terms.* Thus, the 11th term is $2+(10\times4)$, or 42, and the 20th term is $2+(19\times4)$, or 78. Again, in the series 100, 93, 86, 79, the same rule applies, except that we

must subtract from the first term, instead of adding to it. Thus the 6th term is $100-(7\times5)$, or 65.

The rule by which we find the sum of an arithmetical series is easily discovered, if we write the series down, and beneath it place the same series in reverse order, and add corresponding terms. Thus, in the case of 7 terms of the series 5, 11, 17, &c.:—

$$\begin{array}{ll} \text{Sum of the series} & = 5+11+17+23+29+35+41 \\ \text{Also sum of the series} & = \underline{41+35+29+23+17+11+5} \\ \text{Therefore twice the sum} & = 46+46+46+46+46+46+46 \end{array}$$

Here the sum of the first and last terms is 46, and the sum of each pair of terms being the same, it follows that twice the sum of the series is equal to 46×7, or 322, and the sum of the series is therefore 161. From this and similar instances, we may deduce the general rule, that *the sum of an arithmetical series is half the sum of the first and last terms multiplied by the number of terms.*

Find the 13th term and the sum of 13 terms of the series 3, 11, 19, &c. Here 13th term $=3+(12\times8)=99$. Sum of 13 terms $=\frac{1}{2}\times(99+3)\times13=663$.

Find the sum of 21 terms of the series 78, 75, 72, &c.

Here 21st term $=78-(20\times3)=18$. Sum of 21 terms $=\frac{1}{2}\times(78+18)\times21=1008$.

A man undertakes for a wager to walk $\frac{1}{4}$ of a mile in the first hour, $\frac{1}{2}$ a mile in the second, $\frac{3}{4}$ of a mile in the third, and so on, for 18 hours. How many miles will he walk in the last hour, and how many altogether? In the 18th hour he will walk $\frac{1}{4}+(17\times\frac{1}{4})=4\frac{1}{2}$ miles, and altogether he will walk $\frac{1}{2}\times(4\frac{1}{2}+\frac{1}{4})\times18=\frac{1}{2}\times\frac{19}{4}\times\frac{18}{1}=\frac{171}{4}=42\frac{3}{4}$ miles.

A series of numbers such that each is equal to the product of the one before it, and some fixed quantity, is called a *Geometrical Series*, and the numbers themselves are said to be in Geometrical Progression. Thus, 7, 21, 63, 189, 567, is such a series, the fixed quantity, or *common ratio*, as it is called, being 3. When the first term and the common ratio are known, any term of the series may be easily found. For example, in the series 7, 21, 63, &c., the second

term is equal to the first multiplied by 3, the third term is equal to the first multiplied by 9, or 3^2, the fourth is equal to the first multiplied by 27, or 3^3, and so on, *any term being equal to the product of the first term and the common ratio raised to a power one less than the number of terms.* Thus, the 9th term of the series is $7 \times 3^8 = 45927$. Again, the 12th term of the series, 56, 28, 14, &c., where the common ratio is $\frac{1}{2}$, is $56 \times (\frac{1}{2})^{11} = 56 \times \frac{1}{2048} = \frac{7}{256}$.

The rule by which we find the sum of a geometrical series is easily discovered, if we write the series down, and beneath it place a series formed by multiplying all the terms by the common ratio, and subtract the upper line from the lower. Thus, in the case of 5 terms of the series 7, 21, 63, &c.:—

$$\text{Sum of the series} = 7 + 21 + 63 + 189 + 567$$
$$3 \times \text{sum of the series} = 21 + 63 + 189 + 567 + 1701$$

Hence $2 \times$ sum of the series $= 1701 - 7$, or 1694; for all the other terms, being the same in both lines, disappear in subtraction, and therefore the sum of the series is $\frac{1}{2} \times 1694$, or 847. From this and similar instances, we may deduce the general rule, *carry on the series to one more term, and divide the difference between this last found term and the first term by the difference between the common ratio and unity.*

Find the 8th term and the sum of 8 terms of the series 11, 22, 44, &c. Here 8th term $= 11 \times 2^7 = 11 \times 128 = 1408$; the 9th term will consequently be 2416, and the sum of 8 terms $= (2416 - 11) \div (2 - 1) = 2405$.

Find the sum of 6 terms of the series 128, 32, 8, &c.

Here the 7th term $= 128 \times (\frac{1}{4})^6 = \frac{128}{1} \times \frac{1}{4096} = \frac{1}{32}$. Therefore the sum of 6 terms $= (128 - \frac{1}{32}) \div (1 - \frac{1}{4}) = \frac{4095}{32} \times \frac{4}{3} = \frac{1365}{8} = 170\frac{5}{8}$. (Note M.)

Ex. 43.

Find—

1. The 10th term of the A. S. 2, 7, 12, 17, &c.
2. The 13th term of the A. S. 1095, 1032, 969, &c.
3. The sum of 8 terms of the A. S. 6, 14, 22, &c.
4. The sum of 10 terms of the A. S. 30, 27, 24, &c.

5. The 9th term and the sum of 12 terms of the G. S. 6, 12, 24, &c.

6. The 7th term and the sum of 5 terms of the G. S. 36, 12, 4, &c.

7. The 8th term of an A. S. is $31\frac{1}{2}$, and the 5th term is 18. Find the sum of 10 terms.

8. The 3rd term of a G. S. is $3\frac{1}{5}$, and the 6th term is $1\frac{399}{625}$. Find the 4th term.

9. A body falling by the action of gravity, falls 16 feet in the first second, 48 feet in the second second, 80 feet in the third second, and so on in A. P. Through what distance will it fall in 10 seconds?

10. Which is the cheaper, and by how much, to buy 12 yards of point lace at £5 5*s*. per yard, or to pay one farthing for the first yard, 3 farthings for the second, 9 for the third, and so on in G. P.?

Some account of the nature and use of *Logarithms* will now be given, but it will be first necessary to make extensions in the meanings to be attached to certain signs, of a similar nature to those already made with respect to the signs of multiplication and division. In the first place, then, we know from ordinary subtraction, that $8-6=2$, or, when expressed in words, that the result of subtracting 6 from 8 is 2. This is an instance of subtracting a less number from a greater. Let us now consider what would be the result of subtracting a greater number from a less; for example, 10 from 8. One thing is certain from the effect of the laws of addition and subtraction already laid down, namely, that the result of taking away 10 from any number is the same thing as taking away 8, and afterwards taking away 2 more. Thus, $50-10=50-8-2$, each being 40, and similarly, whatever $8-10$ may mean, it must be the same as $8-8-2$, or $0-2$. A similar expression is obtained in any instance of subtracting a greater number from a less. Thus, $29-36=29-29-7=0-7$; $54-72=54-54-18=0-18$. Beyond this point we cannot go, and $0-18$, or leaving out the cipher, as something to be understood, -18, is the simplest form in which we can express the result of taking 72 from 54. Expressions, such as -18, -7, -2, are said to signify *negative quantities*. An idea of their meaning is best obtained by considering the

case of debts. If a man has £54, and is obliged to pay £72, he is worse off than if he possess nothing, for after his £54 has gone in part payment, £18 remains to be discharged out of any future assets. He may be said then to possess —£18. When we have once recognised the existence of these negative quantities, we ought to be able to add or subtract them, and to multiply or divide by them just as we do with ordinary numbers, which we may call, in contradistinction to the others, *positive quantities.* When viewed in this light, positive quantities are supposed to have the sign + either expressed or understood, so that 14, 19, 20, are abbreviations for $0+14$, $0+19$, $0+20$, just as -14, -19, -20, are abbreviations for $0-14$, $0-19$, $0-20$.

Now, in addition and subtraction, when confined to such cases as gave positive results, we found that the order of the numbers was immaterial. Thus, $26-5+4-2$, and $26+4-2-5$, and $26-2-5+4$, all give the same result, 23. Extending this law to the case of negative quantities, we find that -7 added to 11, must be the same as 11 added to -7, and must therefore be $0-7+11$, or $0+11-7$, or 4. Hence, *adding a negative quantity is the same thing as subtracting a positive quantity.*

If this rule be applied to the case of several numbers, positive and negative, all added together, it is best to take the sums of the numbers with each sign, and the difference of these results, with the sign of the greater prefixed, will be the value of the whole expression.

Thus $18+20-51+17-108+49=104-159=-55$

$27-31+62-14+8-22=97-67=30.$

Next, to subtract a negative quantity, as, for example, —4, from another quantity. Now -4 is only an abbreviation for $0-4$, the remainder when 4 is taken from 0. Let us, therefore, examine the case of taking away the remainder arising from a possible subtraction, and let it lead us to the result in the case of a negative remainder. We

know that $13-9=4$, and if 4 be taken from 47, the result is 43. This number may also be obtained from 47, by adding the 9 and subtracting the 13, or expressing by signs $47-4=47+9-13$. From this and similar instances, deriving the general law that subtracting a remainder is equivalent to adding the number taken away, and subtracting the number from which it is taken away, we must infer that since $0-4=-4$, therefore $12-(-4)=12+4-0=16$. *Hence subtracting a negative quantity is the same thing as adding a positive quantity.* Thus, $8-(-6)=14$, $-11-(-9)=-2$.

The truth of this and the preceding rule may be illustrated by the case of debts, which, as has been stated, may be regarded as negative property. If we add on debts to a man's estate we diminish his property, if we take away debts we increase it.

Next, to multiply by a negative quantity. We found in ordinary multiplication, that the sum of the product of a number by several multipliers was equal to the product of the number and the sum of the multipliers. Thus, $(8\times7)+(8\times10)=56+80=136=8\times17$. Extending this rule to negative quantities, it follows, that $5\times(-3)$ added to (5×3) must give $5\times(3-3)$, or 5×0, which is 0. Hence $5\times(-3)+15=0$, and therefore $5\times(-3)$ must be -15. And evidently, also, $(-3)\times5=-15$.

Next, take the case of $(-3)\times(-5)$. Now $(-3)\times(-5)+(-3)\times5=(-3)\times$ the sum of 5 and (-5), that is, -3×0, or 0. And we know that $(-3)\times5=-15$. Therefore $(-3)\times(-5)-15=0$, and, consequently, $(-3)\times(-5)$ must be 15.

Collecting together all the possible combinations of positive and negative quantities, we have—

$$3\times5=15 \qquad (-3)\times5=-15 \qquad 3\times(-5)=-15 \qquad (-3)\times(-5)=15.$$

Hence, from multiplication and division being contrary operations, we have—

$15 \div 3 = 5 \quad (-15) \div (-3) = 5 \quad (-15) \div 3 = -5 \quad 15 \div (-3) = -5.$

Thus we derive the general rule, applicable to both multiplication and division, that *like signs give* $+$, *and unlike signs give* $-$.

The following are examples of the application of the above principle :—

$$\frac{-6}{-2} + 5 - 4 + \frac{72}{-9} = 3 + 5 - 4 - 8 = -4.$$

$$\frac{7+3-8-12}{24-29} \times \frac{43-36+50}{11-30} = \frac{-10}{-5} \times \frac{57}{-19} = 2 \times -3 = -6.$$

$$(-2) \times (-3) \times (-4) \times (-5) = 120.$$

$$(-1) \times (-2) \times (-3) \times (-4) \times (-5) = -120.$$

Another extension of meaning remains to be noticed before logarithms can be explained. It will be remembered that 2^4 was an abbreviation for $2 \times 2 \times 2 \times 2$, 5^3 for $5 \times 5 \times 5$, 7^{20} for the continued product of 7 repeated 20 times, and that the numbers 4, 3, 20, placed above the others, were called *indices*. Now, such expressions as $8^{\frac{2}{3}}$, 2^0, 11^{-3}, may be written, and it may be asked whether any meaning can be assigned to them. One thing we settle beforehand, namely, that whatever these meanings may be, the same laws shall hold good as held good in the case of positive indices. Now, since 2^{12} is an abbreviation for the continued product, or, more shortly, the C. P. of 2 12 times repeated, and 2^7 for the C. P. of 2 7 times repeated, it follows that $2^{12} \times 2^7$ must be the C. P. of 2 19 times repeated, or $2^{12} \times 2^7 = 2^{19}$. In this way it is seen that *the product of different powers of the same quantity is obtained by adding their indices.* Again, $2^{12} \div 2^7$ must be the C. P. of 2 repeated five times, as seven factors are struck out of the original twelve. Therefore $2^{12} \div 2^7 = 2^5$. Hence *to divide a power of any quantity by another power of the same, subtract the second index from the first.*

It follows, from the above, that $2^4 \times 2^4 \times 2^4 \times 2^4 \times 2^4$ equals 2^{20}, and hence, *to raise a power of a quantity to any power,*

multiply the index by the number of the power to which it is to be raised.

Since 2^{20} is the fifth power of 2^4, and it has been explained (page 152) that power and root are correlative terms, it follows that 2^4 is the fifth root of 2^{20}; hence, *to find any root of a power of a quantity, divide the index by the number of the root.*

The following are illustrations of the preceding rules:—

$$2^5 \times 2^{11} = 2^{16} \qquad 4^5 \div 4 = 4^4 \qquad 6^{11} \div 6^3 = 6^8 \qquad 10^{17} \times 10^5 = 10^{22}$$
$$(3^2)^8 = 3^{16} \qquad (4^5)^2 = 4^{10} \qquad \sqrt{12^8} = 12^4 \qquad \sqrt[5]{12^{15}} = 12^3.$$

To make these illustrations clear, it should be explained, that a number occurring without any index is supposed to have the index 1 understood. Thus, $4^5 \div 4$ is the same as $4^5 \div 4^1$. Also, $\sqrt[3]{\ }$ $\sqrt[4]{\ }$ are abbreviations for the 'cube root of,' the 'fourth root of,' and so on. The sign $\sqrt{\ }$, without any number, is understood to mean $\sqrt[2]{\ }$, or the 'square root of.'

We have now sufficient materials for determining the significations of $2^{\frac{3}{4}}$, 2^0, 2^{-7}. If we choose to adhere to our definition that the index means the number of times the factor occurs, no meanings can be given to these expressions, as in that case they would be the C. P. of 2 repeated three-quarters of a time, no times at all, and seven less than nothing times respectively. But if, on the contrary, we choose to adhere to the laws just found, and say that they shall be applicable to all cases, whether the indices be integral or fractional, positive or negative, we shall then at once determine the meanings of the given expressions. First, therefore, in the case of $2^{\frac{3}{4}}$. Let it be raised to the power of 4. Then $2^{\frac{3}{4}}$ to the fourth power must be, by multiplying the index, 2^3. Hence 2^3 is the fourth power of $2^{\frac{3}{4}}$, and consequently $2^{\frac{3}{4}}$ is the fourth root of 2^3 or $\sqrt[4]{2^3}$. Next, to find the meaning of 2^0. Multiply it by any positive power of 2, say 2^3, then $2^0 \times 2^3$ must be, by adding the

indices, 2^3. Hence 2^3 is the product of 2^3 and 2^0, and therefore $2^0=1$. It would be similarly found that any number raised to the power of 0 is equal to 1. (Note N.) Lastly, to find the meaning of 2^{-7}. Multiply it by 2^7, and we have $2^{-7}\times 2^7=2^0=1$. Hence $2^{-7}=\frac{1}{2^7}$.

The following are instances of negative and fractional indices :—

$$107^{\frac{3}{5}}=\sqrt[5]{107^3} \qquad 96^{\frac{2}{3}}=\sqrt[3]{96^2} \qquad 179^0=1 \qquad 13^{-11}=\frac{1}{13^{11}}$$

The sole object of the above explanations is to make an account of logarithms intelligible. In order to treat this subject thoroughly, some knowledge of Algebra is absolutely necessary, and it must, therefore, not be supposed that a complete account is here attempted. It is hoped that enough may be given to enable the reader to understand what logarithms are, and what is their use ; but the methods of computing them, their relations to geometry, and many other particulars respecting them, are only to be understood when some progress in mathematics has been accomplished. No mention will be here made of any other system of logarithms than that in ordinary use, so that all definitions and explanations must be understood as applying exclusively to the common system.

If n and N be two numerical quantities so related that $10^n=N$, then n is called the logarithm of N, or, shortly, $n=\log N$. From this we see at once, that since $10^3=1000$; $10^2=100$; $10^1=10$; $10^0=1$; $10^{-1}=\frac{1}{10}$; $10^{-2}=\frac{1}{100}$; $10^{-3}=\frac{1}{1000}$; therefore $\log 1000=3$; $\log 100=2$; $\log 10=1$; $\log 1=0$; $\log \frac{1}{10}$, or, when expressed decimally, $\log \cdot 1=-1$; $\log \cdot 01=-2$; $\log \cdot 001=-3$; and this list might be extended to any integral, positive, or negative powers of 10 whatever.

As has been stated, no attempt will be made to show how tables of logarithms are computed; but on looking over a

table, it will be evident that the logarithms of numbers gradually increase as the numbers increase. Thus the logarithm of any number between 10 and 100 will be somewhere between 1, the logarithm of 10, and 2, that of 100. Hence any number of two figures will have for its logarithm, 1 followed by some decimal. For example, log 37= 1·5682017. Similarly, any number of three figures being between 100 and 1000, its logarithm must be 2 and a decimal. Thus, log 829=2·9185545. The integral part of a logarithm is called its *characteristic*, and the decimal part is called its *mantissa*. Hence, from the above instances, we may deduce the rule that *the characteristic of the logarithm of any numerical quantity is* 1 *less than the number of integral figures in the quantity.*

Thus, log 557·9=2·7465564, log 8479·628=3·9283769. From these logarithms we may deduce those of any quantities having the same significant figures. For since 557·9 $= 10^{2 \cdot 7465564}$, therefore $55790 = 10^{2 \cdot 7465564} \times 10^2 = 10^{4 \cdot 7465564}$. Hence 4·7465564 = log 55790. Again, $\cdot 005579 = 557 \cdot 9 \div 100000 = 10^{2 \cdot 7465564} \div 10^5$, and therefore the logarithm of ·005579 must be 2·7465564−5, or −2·2534435. This is the same as −3+·7465564, and we may therefore consider the logarithm composed of a negative characteristic 3, and a positive mantissa ·7465564. To express this, the negative sign is written above the characteristic, thus, $\bar{3} \cdot 7465564$. Hence the logarithms of 557·9, 55790, and ·005579, all have the same mantissa, ·7465564, and their respective characteristics are 2, 4, and −3. From this and similar instances we deduce the general rule, that *the mantissa is the same for all numerical quantities having the same significant figures, that the characteristic is* 1 *less than the number of integral figures, and where there are no integral figures, it is negative, and* 1 *more than the number of ciphers after the decimal point.* The following instances may serve to explain this more clearly:—

Log 24095=4·3819269. Log 240·95=2·3819269.

Log 24095000=7·3819269. Log ·000024095 = $\bar{5} \cdot 3819269$.

In ordinary tables of logarithms the mantissa alone is ven, and is calculated generally to seven places of deci-als, and for all combinations of five figures, from 10000 to 999.

One of the most convenient tables for general use is that iblished by Messrs. Chambers, in a small volume, price . 6*d.*, which contains, in addition, the most useful trigo-ometrical and other mathematical tables.

Having thus explained what logarithms are, it remains show what is their use. First, then, let m and n be the garithms of M and N, then $10^m = M$ and $10^n = N$. Conse-iently $M \times N = 10^{m+n}$, and therefore $m + n = \log (M \times N)$. [ence, in multiplication, *the sum of the logarithms of the ultiplier and multiplicand is the logarithm of the product.* 'hus multiplication is done by addition.

Again $M \div N = 10^{m-n}$, and therefore $m - n = \log (M \div N)$. [ence, in division, *the logarithm of the divisor, subtracted om that of the dividend, leaves as remainder the logarithm f the quotient.* Thus division is done by subtraction.

Again, $M^k = 10^{km}$, and therefore $k\,m = \log (M^k)$. Hence *the garithm of a power of a quantity is the logarithm of the uantity multiplied by the number of the power.* Thus aising to powers is done by multiplication.

Lastly, $\sqrt[k]{M} = M^{\frac{1}{k}} = 10^{\frac{m}{k}}$, and therefore $\frac{m}{k} = \log \sqrt[k]{M}$. [ence *the logarithm of a root of a quantity is the logarithm f the quantity divided by the number of the root.* Thus xtraction of roots is done by division.

It must be noticed, that we do not, in general, by loga-ithms obtain exact answers, but only such as are close pproximations to the truth. In such a table of logarithms s has been described, the greatest error that could occur ould be 1 in the 6th significant figure; the greatest pos-ible effect of which would be to make the result wrong by $\frac{1}{1000}$ per cent.

The following are examples adapted for logarithmic omputation:—

Divide 7·4893 by 82·765.

$$\begin{array}{rl} \text{Here } \log 7{\cdot}4893 = & {\cdot}8744412 \\ \log 82{\cdot}765 = & 1{\cdot}9178467 \\ \hline & \bar{2}{\cdot}9565945 = \log {\cdot}0904887 \end{array} \qquad \text{Result} = {\cdot}0904887.$$

Find the continued product of 7·8653, 216·49, and ·0352, and divide it by ·189.

$$\begin{array}{rl} \text{Here } \log 7{\cdot}8653 = & {\cdot}8957153 \\ \log 216{\cdot}49 = & 2{\cdot}3354378 \\ \log {\cdot}0352 = & \bar{2}{\cdot}5465427 \\ \hline & 1{\cdot}7776958 = \log 59{\cdot}9371 \\ \log {\cdot}189 = & \bar{1}{\cdot}2764618 \\ \hline & 2{\cdot}5012340 = \log 317{\cdot}1276 \end{array}$$

Therefore the results are 59·9371 and 317·1276.

In how many years will any principal increase to five times its amount at 4 per cent. compound interest?

Any principal P will in T years at 4 per cent. amount to $P \times (1{\cdot}04)^T$, and hence from the conditions of the question $(1{\cdot}04)^T$ must be equal to 5. Hence $T \times \log (1{\cdot}04)$ must be equal to log 5, and therefore $T = \log 5 \div \log (1{\cdot}04) = {\cdot}6989700 \div {\cdot}0170333$, or about 41·0355 years. (Note O.)

Find the present value of an annuity of £200, to commence in 5 years and to continue 15, on the supposition of compound interest at 4 per cent.

Here the present value of the first annual payment will be £200 $\div (1{\cdot}04)^5$; of the second, £200 $\div (1{\cdot}04)^6$; of the third, £200 $\div (1{\cdot}04)^7$; and so on, these present values being £200 successively multiplied by the terms of a geometrical series, of which the first term is $1 \div (1{\cdot}04)^5$, and the common ratio is $1 \div (1{\cdot}04)$. Hence the 16th term is $1 \div (1{\cdot}04)^{20}$. Applying logarithms to calculate the value of this expression we find it to be ·456388, and the value of the 1st term is ·821927. The sum of 15 terms of the series will consequently be ·821927 − ·456388, divided by the difference between 1 and $(1 \div 1{\cdot}04)$, or 1 − ·961538 or ·038462. The sum of the series is therefore ·365539 ÷ ·038462, or, as may be found by logarithms, 95·039; and hence the value of the deferred annuity is this quantity multiplied by £200, that is, £1900·78, or £1900 15*s.* 7*d.* nearly.

Ex. 44.

Find, by logarithms, the values of—

1. $874{\cdot}372 \times {\cdot}05361$.
2. $7298{\cdot}1 \times {\cdot}00326$.
3. $43879 \div 675980$.
4. $27{\cdot}635 \times 81{\cdot}257 \times {\cdot}0003762 \times 530000$.

5. The 16th term of the G. S. 8·8, 9·68, 10·648, &c.

6. The sum of 22 terms of the G. S. ·012, ·144, ·1728, &c.

7. What will be the amount of £577 13s. 8d. for 7 years, at 6 per cent. compound interest?

8. What will be the compound interest on £1047 16s. 2d. for 13 years, at $3\frac{1}{2}$ per cent.?

9. What is the present value of £899 17s. $2\frac{1}{4}d.$, due 5 years hence, at 4 per cent. compound interest?

10. In how many years will any principal increase to six times its amount, at 5 per cent. compound interest?

11. A man saves £20 a year, and puts it into a bank where 5 per cent. compound interest is allowed. At the end of 20 years from the date of his first deposit, he draws out all his money. How much does he receive?

12. Find the value of an annuity of £300, to commence in 7 years, and to continue 11 years, at 4 per cent. compound interest.

MISCELLANEOUS EXAMPLES.

Throughout the following examples, unless anything be stated to the contrary, the following particulars are assumed:—

1. The ratio of the diameter to the circumference of a circle = 113 : 355.
2. The ratio of the area of a circle to that of the circumscribing square = 355 : 455.
3. A cubic foot of water weighs 1000 ounces avoirdupois.
4. 11 gallons contain 3050 cubic inches.

It is not intended that logarithms should be used in any of the examples.

An asterisk prefixed to a question means that an exact answer is not required.

---

1. Find the value of 14 acres, 1 rood, 12 perches, at £5 7s. 6d. per acre.

2. An advertisement costs 18s. for insertion in a paper where the charge is 3d. a line, which on the average contains 10 words. What will be the cost of inserting it in a paper where the charge is 6d. per line of 12 words?

3. Find the simple interest on £850 for $2\frac{1}{2}$ years at $6\frac{2}{3}$ per cent.

4. If the tax on wine and spirits were increased 50 per cent., and in

consequence the amount imported were diminished 50 per cent., what would the Government gain or lose per cent. by the change?

5. What dividend would result from investing £1560 in the 3 per cents. at $81\frac{1}{4}$?

6. Find the square root of 2832489 and of 36·343 to four places of decimals.

7. Divide 2136 among 4 persons, so that their shares may be proportional to the numbers 32, 26, 23, and 15.

8. At what rate per cent. will £184 amount to £213 18*s.* in $3\frac{1}{4}$ years?

9. Add together the fractions $\frac{7}{11}$, $8\frac{3}{8}$, $2\frac{7}{10}$, $\frac{13}{15}$, and $4\frac{9}{23}$; find the G.C.M. of 41377 and 62790; the L.C.M. of 8, 18, 48, 108; the value of $\frac{13}{37}$ of a ton, and the decimal it is of 1 lb.

10. At what time between 4 and 5 will the hour and minute hands of a clock be together, and at what time will they be opposite?

11. If a coach wheel makes 64 revolutions per minute, while the coach is going 8 miles an hour, find its circumference.

12. Find the discount on a bill for £139 18*s.* 4*d.* drawn August 10 at 60 days, and discounted September 17 at $7\frac{1}{2}$ per cent.

13. Newton's 'fusible alloy' is composed of 8 parts of bismuth, 5 of lead, and 3 of tin. What weight of each is there in 1 cwt. of the compound?

14. Multiply ·00875 by 2·832; divide ·0476 by 21250; and add together 7·025, $2\frac{1}{3}$, and $6\frac{3}{11}$, expressing the result as a decimal.

15. A can do a piece of work in 13 days, which B can do in 17. In what time would they do it together?

16. Find the income arising from investing £1150 1*s.* in the 3 per cents. at $92\frac{1}{4}$.

17. Find the square root of 4533194241, and the cube root of 5851·461608.

18. If ·2555 of $\frac{16}{73}$ of 2 qr. $6\frac{1}{2}$ lb. of coal cost ·0328125 of a shilling, what is the price per ton?

19. Find the sum of 9 terms of the arithmetical series 8, $10\frac{2}{3}$, $13\frac{1}{3}$, &c., and the sum of 6 terms of the geometrical series 100, 80, 64, &c.

20. Eggs being bought at 5*d.* per dozen, how many can be sold for a shilling, so as to gain 20 per cent.?

21. A party of 3 men undertake to clear timber on a road through the bush, at 15*s.* per chain, and after having been engaged for 50 working days, they find it necessary to take on a fourth man, in order that the distance of 2 miles 56 chains may be finished in the specified time of 80 days. What are the average earnings of each man per day?

22. Supposing the duty on two kinds of foreign wine to be 7*s.* and 4*s.* per gallon, and that, by an alteration in the law, an uniform duty of

5*s.* was substituted, and consequently the consumption of the one increased 15 per cent., and of the other decreased 25 per cent., while the cost of collection was diminished from 4*d.* to 3*d.* per gallon, the quantity imported being at first 20,000 and 70,000 gallons respectively, how much was imported after the alteration of the law, and what was the gain or loss to the revenue?

23. Find (1) the Simple Interest, (2) the Compound Interest, (3) the true Discount on £390 10*s.* for 2 years at 5 per cent.

24. How long will it take for a train 280 feet long, going at the rate of 20 miles an hour, to pass another 160 feet long coming in the opposite direction at 40 miles an hour?

25. A, B, C, and D invest capital in a manufactory to the extent of £400, £600, £800, and £1000 respectively. The profit at the end of the year being £350, how much ought each to receive?

26. Add together $4\frac{3}{5}$, $2\frac{2}{3}$, $\frac{9}{16}$, $6\frac{11}{45}$; reduce $\frac{107}{625}$ and $\frac{394}{1025}$ to decimals; multiply $\cdot01\dot{8}\dot{1}$ by $29\cdot4\dot{9}6\dot{2}$; and divide $\cdot0168$ by $2\frac{3}{11}$.

27. If an income of £36 1*s.* is derived from investing £1017 2*s.* 6*d.* in $3\frac{1}{2}$ per cent. stock, at what price is the stock bought?

28. In a seam of coal 8 feet thick, 12584 tons are contained in an acre. What is the specific gravity of coal, that is, the ratio between its weight and that of water?

29. Separate into their prime factors the numbers 165672, 165025, and 71706.

30. If a certain sum of money will pay A's and B's wages for $3\frac{3}{8}$ days, or A's only for 6 days, for what time would it pay B's wages?

*31. Find the square root of 136·7 and the cube root of $\frac{9}{11}$.

32. If $4\frac{2}{11}$ of 3 cwt. 3 qr. $3\frac{1}{2}$ lb. cost £12 14*s.* 11*d.*, what will be the price of 1 qr. 16 lb.

33. Supposing the Dun Mountain Railway to have a rise of 1 in 20 for $\frac{1}{5}$ of its length, 1 in 30 for half its length, 1 in 25 for $\frac{1}{10}$, and the rest to be level, what elevation would it have reached in $13\frac{7}{11}$ miles?

34. Divide 655 guineas among 4 persons in the proportion of the numbers $2\frac{1}{2}$, $3\frac{1}{3}$, $4\frac{1}{4}$, $5\frac{1}{5}$.

35. Find (i) the simple interest on £447 15*s.* for 4 years at $2\frac{1}{2}$ per cent., (ii) the amount of £136 2*s.* 11*d.* for 2 years at 4 per cent. compound interest, (iii) the true discount on £188 6*s.* 8*d.* for $1\frac{1}{3}$ years at 6 per cent.

36. Find both the G. C. M. and the L. C. M. of the numbers 4320, 5472, 6048.

37. A man buys a number of yards of cloth, and a quarter of them having been stolen, sells the remainder at 5*s.* 6*d.* per yard, thereby gaining $3\frac{1}{8}$ per cent. on the whole transaction. His actual gain being 18*s.* 9*d.*, how many yards did he buy?

38. Find the value of $\frac{15}{176}$ of 2 cwt. 3 qr.; reduce 1 dwt. 11 grains to the decimal of 1 lb. avoirdupois, and subtract $2\cdot04761\dot{9}$ of 2 guineas from $1\cdot0844\dot{5}\dot{9}$ of £18 10*s*.

39. A Prussian thaler containing 30 silber groschen is equivalent to 3*s*. English. Express £20 17*s*. $10\frac{1}{2}$*d*. in Prussian money.

40. If $\cdot918$ of £15 12*s*. 6*d*. will pay for a carpet $\frac{17}{25}$ of $9\frac{1}{11}$ of 6 yards long, and $\cdot1\dot{6}$ of $4\frac{1}{8}$ yards wide, how much is that per square foot?

41. A man on horseback leaves Westminster for Putney and Richmond at 2 o'clock, and a man on foot leaves Putney for Richmond at 1.30. The former, who rides at the rate of $6\frac{1}{2}$ miles an hour, comes up with the latter, who walks at the rate of $3\frac{1}{2}$ miles, at 5 min. past 4. Find the distance from Westminster to Putney.

42. A cubic foot of elm weighs 42 lb., and 80 planks, each 16 feet long and 9 inches broad, weigh 1·125 tons. Find the thickness of the planks.

43. A and B together can do a piece of work in 48 days, and A and C together can do it in 44. Supposing B's and C's rates of working to be as 4 : 5, in what time would they do it together?

44. A boy comes home from school three times in the year, and his father incautiously promises him a farthing as pocket-money for the first holidays, a halfpenny for the second, a penny for the third, and so on in geometrical progression. If he stay at school 7 years, thus coming home 20 times, what would he receive in his last holidays, and what would he have received altogether?

45. Find the square roots of the sums of the cubes of the first ten numbers, eleven numbers, and twelve numbers.

*46. In 'The Times' of Nov. 14, 1863, a correspondent states that in Ireland 5,661179 acres are used for growing crops, 9,719955 are grass, 39441 fallow, 318760 woods and plantations, and 4,580589 are bogs and waste land. Find as far as two places of decimals what percentage each class of land is of the whole area of the country.

47. Out of a basket of pears and plums there are 8 pears together worth one shilling. The basket itself is worth as much as $1\frac{3}{8}$ plums, or $1\frac{1}{3}$ pears. Find the price of a dozen plums.

48. Find the compound interest on £632 10*s*. for 3 years at 5 per cent.

49. How many prime numbers are there between 500 and 525?

50. The cost price of candles being 10*d*. per lb., the loss from a reduction of $\frac{1}{2}$*d*. in the selling price would be just made up by an increase of 20 per cent. in the quantity sold. What is the present selling price?

51. At what rate per cent. would £1000 amount to £1071 4*s*. 6*d*. in 2 years at compound interest?

52. In the House of Commons, 16 seats being vacant from various

causes, ·0375 of the remaining members voted against the Government, and $\frac{17}{20}$ of $1\frac{1}{16}$ were not at the division. The majority being 14, what is the full number of members of the House?

*53. The ratio of the diameter to the circumference of a circle is 1 : 3·1415926536 true to 10 places of decimals. Find the difference between this decimal and the fraction $\frac{113}{355}$, and find to the nearest inch the error that would arise from taking the latter ratio in place of the former, when calculating the circumference of a circle 8000 miles in diameter.

54. A sum of £2752 15*s.* is invested in the 3 per cent. consols at $90\frac{3}{4}$, is afterwards sold out at $91\frac{1}{2}$ and the proceeds invested in an Australian 5 per cent. loan at 105. Find the difference of income produced by the transfer.

55. At what times between 5 and 6 will the hands of a clock be at right angles to one another?

56. Find the present value of £1954 13*s.* 9*d.* due $2\frac{1}{2}$ years hence at 5 per cent. simple interest.

*57. The weight of a cubic inch of water is 252·458 grains. Find the per-centage of error that arises from considering a cubic foot as weighing 1000 ounces.

*58. Find how long a man walking 3 miles an hour will take to walk round a circular enclosure containing 50 acres.

59. £495 is divided among A, B, C, and D. A's share is $\frac{3}{4}$ of B's, $\frac{2}{3}$ of C's, and £60 less than D's. Find the share of each.

60. A cistern can be filled by one of two pipes in 20 minutes, and by the other in 30. They are both opened together, but an obstruction in the second causes it for part of the time to discharge only $\frac{3}{5}$ of the usual quantity of water. If the time after removing the obstruction is three times that before removing it, how long has altogether been taken to fill the cistern?

61. What is the present value of £1464 13*s.* 4*d.* due 3 years hence at 4 per cent. compound interest?

62. Between 1500 and 2000 how many numbers are there whose least divisor is 19?

63. In the question No. 20, Ex. 13, find whether the three men would ever be together anywhere but at the starting-point, and, if so, at what times?

64. A man rows a distance of $6\frac{1}{2}$ miles in 2 hours 36 min. against a stream running at the rate of $2\frac{1}{2}$ miles an hour. How long would he have taken to row the same distance with the stream?

65. A cubic foot of iron weighs 480 lb. What is the length of the edge of a cubical block of iron weighing ·0735 of a ton?

66. Find the 18th term and sum of 27 terms of the arithmetical

progression 7, 12, 17, and the 5th term and sum of 7 terms of the geometrical progression, 11, 33, 99, &c.

67. If the true discount on a certain sum were £2 17*s.* 5*d.*, interest being at 5 per cent., what would it be if interest were at 6 per cent.?

68. By transferring from the 3 per cents. to the $3\frac{1}{4}$ per cents., a man increases his income in the ratio of 169 : 168. The price of the former stock being 91, what is the price of the latter?

69. If 15 oz. of gold, 20 carats fine, be mixed with 9 oz. 18 carats fine, and the mixture afterwards refined down to 21 carats fine, what will then be its weight?

70. If 20 per cent. is gained by selling walnuts at 6*d.* per 100, what would be gained by selling them at 14 for a penny?

*71. Find the 6th root of 18.

72. The difference in the incomes arising from investing a certain sum in the 3 per cents. at 92, or the $3\frac{1}{2}$ per cents. at $107\frac{1}{4}$ is 1*s.* 8*d.* Find the sum to be invested?

73. If A, B, and C can do a piece of work in 8, 10, and 12 days respectively, in what time would A and B do it with C's help for half the time?

74. The true discount on a bill for £218 13*s.* 4*d.* is £5 6*s.* 8*d.* What does a banker gain by charging common discount?

75. A father leaves to his six sons shares of his property, which decrease in geometrical progression from that of the eldest to that of the youngest. If the shares of the second and fifth are £1782 and £528 respectively, find that of the eldest, and the amount left altogether.

76. Two boats start at 12 o'clock for a race. At 5 minutes past 12 the losing boat is 832 yards from the winning flag, and at 8 minutes past 12 the winning boat comes in 64 yards a-head. Find the length of the course, and the speed of each boat in miles per hour.

77. Divide £14519 10*s.* in the ratio of the quantities 20, 16·125, 15·32$\dot{7}$ and $14\frac{6}{11}$.

78. A cubic inch of air weighs ·31 grains. How many cubic feet must a balloon contain in order that when filled with gas of $\frac{5}{8}$ the weight of air, it may just sustain a total weight of 5 cwt. 3 qr. 7 lb.?

*79. The velocity acquired in falling from a height being proportional to the square root of that height, find what velocity will be acquired in falling 1000 feet, if that due to a fall of 100 feet is 80·225 feet per second.

80. The interest upon £3126 6*s.* 8*d.* being £117 4*s.* 9*d.*, find what would be the true discount for the same time, and at the same rate per cent.

81. Separate into their prime factors the numbers 108680, 48128, and 114954.

82. An increase of 10*d.* in the price per lb. makes a difference of 8 in the rate per cent. of profit. At what price must a cwt. be sold so as to gain 15 per cent.?

83. Two men start at 1 o'clock from the same point, walking in opposite directions at the rate of $3\frac{1}{2}$ and 4 miles an hour respectively. At a quarter past one, a man riding on horseback, at the rate of 7 miles per hour, meets the first. At what time will he overtake the second?

84. Supposing the specific gravity of standard gold to be 18·36, what is its value per oz. troy, if 74137 sovereigns are coined out of $1966\frac{2}{25}$ cubic inches?

85. Find the compound interest on £250 for $2\frac{1}{4}$ years at 5 per cent. per annum.

86. £1 is divided among 4 children proportionally to their ages, which are in arithmetical progression, and altogether amount to 40 years. The eldest gets 4*s.* 6*d.* more than the youngest. What are their ages?

87. A publican pays for brandy, whiskey, and gin, prices in the proportion of 7 : 5 : 3, and sells them at a nominal profit of 30, 25, and 20 per cent. respectively. He, however, increases his profits by adulteration, adding to the brandy 25 per cent. of an inferior spirit, worth only $\frac{1}{5}$ as much, to the whiskey, 20, and to the gin, 10 per cent. of water. His customers consume the three in the proportion of 1 : 3 : 8; his actual gain on the whiskey is 8*s.* 4*d.* per gallon sold, and the amount spent on the spirit for mixing with the brandy £15 8*s.* a year. Find the annual profit on each of the three.

88. By selling a quantity of merino at 2*s.* 6*d.* a yard there is a gain of £1 13*s.* 4*d.*, and by selling it at 2*s.* 2*d.* there is a gain of 30 per cent. How many yards of merino are there?

89. A gentleman delays making an investment in consols, and by so doing gains 3*s.* 2*d.* of income through the funds falling from $91\frac{1}{2}$ to $90\frac{3}{4}$. What was the sum invested?

90. The amount of £3200 at simple interest for 3 years is £3560. What would it be if the interest were compound?

91. Suppose the hour and minute hands of a watch to move in contrary directions, each having its own circle of figures on the dial, the figures in the one circle being placed in contrary order to those in the other. (1) The hands being together at 12 o'clock, how many times will they again be together up to and inclusive of their next meeting at 12? (2) At what time between 3 and 4 will they be together? (3) At what times between 7 and 8 will they be at right angles?

92. Find the number of circulating and non-circulating figures in the decimals equivalent to the fractions $\frac{175}{451}$, $\frac{833}{10504}$, $\frac{145253}{237125}$.

93. A swimming bath, which can be filled by one pipe to a depth of 4 feet in 40 minutes, and emptied by another in 25, is being filled, and the water has a depth of 1 ft. 6 inches, when a boy mischievously opens the waste pipe. Some time afterwards the bath attendant comes in finds only 6 inches of water in the bath, and closes the waste pipe. How long does it altogether take to fill the bath, and, supposing its area to be 61 square yards, how many gallons of water are wasted by the boy's mischief?

*94. Find the diameter of a circular pond 3 feet deep, which contains 10000 gallons of water.

95. A steamboat, 110 ft. in length, starting at 1 o'clock from London up the Thames for Hungerford, meets another of equal size and speed coming in the opposite direction, takes $6\frac{1}{4}$ seconds to pass her, and arrives at Hungerford at 10 minutes past 1. The tide being supposed to be ebbing at the rate of 3 miles an hour, find the distance from London to Hungerford, and the time taken by the second boat to perform the journey.

96. If the compound interest on a sum of money for 2 years at 4 per cent. is £91 16*s*., what would be the true discount on the same sum at the same rate for the same time?

*97. If 20 Austrian florins can be coined from a Cologne mark (3608 grs. troy), and the price of standard silver $\frac{37}{40}$ fine is 5*s*. an oz., what par of exchange in florins per £ is established between London and Vienna? (1 florin = 60 kreutzers.)

98. A dial-plate has three hands, like the hands of a watch, moving upon it. The first goes round in 19 seconds, the second in 22, and the third in 26. If they all begin to move in the same direction from the same point, show that they will again be together at that point in 1 hour 30 minutes 34 seconds; that they will never be together before this, but that some two out of the three will be together 154 times during this period.

99. What is the area of a gravel walk 3 feet broad round a circular grassplat 168 ft. in diameter?

100. Three children are employed in colouring picture-books for a bookseller, and can colour 30, $32\frac{1}{2}$, and $33\frac{3}{4}$ in an hour respectively. While occupied on one parcel they have four times been all three commencing a book at the same moment, and when the parcel is finished, the first and third leave off together, while the second takes between 1 and 2 minutes more to finish the one he is colouring. How many books are there in the parcel?

NOTES.

Note A. Page 4.

In France a somewhat different method of expressing numbers in words is adopted. The two systems agree as far as millions, but a 'billion,' which in England is a million millions, or when expressed in figures 1,000000,000000, is in France only a thousand millions or 1000,000000. Also in the French system a trillion is a thousand billions, a quadrillion is a thousand trillions, and so on; and hence it is found convenient to separate figures into periods of three, instead of periods of six, as we do. It will be a sufficient illustration of the above difference if we separate into periods and express in words the number of arrangements of draughtsmen according to the French mode. It would then be written 677,794,282,450,430,456,394,720, and be read 'Six hundred and seventy-seven sextillions, seven hundred and ninety-four quintillions, two hundred and eighty-two quadrillions, four hundred and fifty trillions, four hundred and thirty billions, four hundred and fifty-six millions, three hundred and ninety-four thousand, seven hundred and twenty.'

Note B. Page 8.

When 10 has been added to the upper figure in order to make subtraction possible, it is more usual to make up for this by adding one to the next figure of the *subtrahend* (or number to be subtracted), than by taking one away from the next figure of the *minuend* (or number to be diminished). The second method is, however, not unfrequently adopted. The two are no doubt equally accurate, but the first is more conveniently applicable to the short form of division mentioned in the next note.

Note C. Page 17.

The form of an example in division may be conveniently abbreviated by omitting to write down the products of the divisor by the successive figures of the quotient, and performing the operations of multiplication and subtraction together. Thus taking the instance already worked out, the shorter form would require the following work to be gone through in the mind. 'Seven times 9 are

```
489)3617456(7397
      1944
       4775
        3746
         323
```

63, 63 from 67 leaves 4; 7 times 8 are 56 and 6 are 62, 62 from 71 leaves 9; 7 times 4 are 28 and 7 are 35, 35 from 36 leaves 1. Bring down the next figure 4; 3 times 9 are 27, 27 from 34 leaves 7; and so on. It is convenient to be able to use this shorter form of Division readily; and although there may be a little difficulty in changing from the one to the other, the increased facility is quite worth the trouble.

Note D. Page 30.

It is easy to say that our tables of weight and measure, and in a less degree our table of money, are inconvenient, but it is difficult to give even a sketch of the numerous questions that have to be considered when any alteration is designed. A help towards doing this will be afforded by first giving an account of the French system, which is in most respects a complete contrast to our own. The standard is entirely different—namely, the length of a quadrant of the earth's meridian, the ten millionth part of which is called the *mètre*, and serves as a basis for what is termed the *metrical system*. The Greek words for ten, hundred, thousand, &c., are used to express multipliers, and the corresponding Latin words are used to express divisors, so that a *decamètre*, *hectomètre*, and *kilomètre* mean 10, 100, and 1000 mètres; while a *decimètre*, *centimètre*, and *millimètre* mean $\frac{1}{10}$, $\frac{1}{100}$, and $\frac{1}{1000}$ of a mètre respectively. The standard for land measure is the *are*, a square decamètre, that for capacity is the *litre*, a cubic decimètre, and that for weight the *gramme*, which is the weight of a cubic centimètre of water under certain defined conditions. The Greek and Latin prefixes are used with these words just as with the word mètre, and thus we speak of a *hectare*, a *decilitre*, or a *kilogramme*. There is much simplicity and consistency in this system, and many persons advocate its adoption in England. It possesses, however, some disadvantages. The standard is one not easily verified. It is very difficult to measure accurately the quadrant of the meridian, and it has been in fact found that the measures from which the length of the mètre was determined were slightly erroneous. On the other hand, the standard is cosmopolitan, showing no preference for any one country over another, whilst the English 'yard' depends upon the length of a seconds' pendulum *in the latitude of London*, and the former is on this account more likely to be universally adopted. Again, the French tables proceed from one denomination to another by the number 10 only; and although this has the great advantage of making concrete numbers and abstract numbers both increase and decrease according to the same scale of local value, still the number 10 is not the most convenient that could have been chosen. Out of all the numbers from 1 to itself only two divide it, namely, 2 and 5, while 12, which is very little greater, has

four factors, 2, 3, 4, and 6. Some persons say, therefore, that the advantage we derive from having 12 so frequently in our tables is greater than that arising from the uniformity of the French system. Others maintain that the great point is to make the same rules and processes do for both abstract and concrete numbers. Others hold that the principal object is to secure uniformity in the weights and measures of different countries, and that other considerations are subordinate. Some are hopeful of success in establishing a decimal system for money, but do not intend to apply it to weights and measures. Others think that the one reform would be useless without the other. Again, a distinction is sometimes drawn between the decimal system and the metrical system, and the former approved of, while the latter is objected to. Lastly, there are those who urge that, before any alteration, we should thoroughly make up our minds as to what we think absolutely best, so that there may be no need to change again; and on the other hand it is contended that our present system is so bad that any tolerably sensible change must be an improvement.

NOTE E. PAGE 32.

The chief authorities on Arithmetic are not in accordance on this point. In Colenso's Arithmetic a quantity less than a penny is expressed as a fraction of a penny, and in that by the Rev. Barnard Smith such a quantity is expressed as farthings and fractions of a farthing, the letter *q* being used in addition to the ordinary *£ s. d.*

NOTE F. PAGE 53.

For a number consisting of more than 3 figures the following test is sometimes useful:—If it be supposed divided into periods of three figures from the right, and the difference between the sums of the odd and even periods is divisible by either 7 or 11 or 13, the number itself is so divisible. Thus 27335 is divisible by 7 and 11, but not by 13, because $335-27$ or 308 is so. Again, 57354281 is not divisible by either of the three, because $354-281-57$ or 16 is not. This test is more apparently than really complicated, and may be applied to many numbers almost at a glance, while it has moreover the advantage of testing for three divisors at once. It is, however, mentioned here mainly as a matter of curiosity.

NOTE G. PAGE 54.

The farthest point to which a separation of numbers into their prime factors has been carried is by a member of the French 'Académie des Sciences,' and of many other scientific bodies, named Burckhardt, who published tables of the least divisors of all numbers in the second and third million. This was intended as a continuation of previous tables

by others, but he finally determined to publish one for the first million, in which the errors in the former tables should be corrected. Copies of the three millions bound in one volume, or of each million separately, are still to be occasionally met with. The first million which, as has been stated, was the last published, has, at the foot of the title page, 'Paris, Mme. V^{e}· Courcier, Imprimateur-Libraire pour les Mathématiques, rue du Jardinet, No. 12. 1817.'

Note H. Page 58.

The short method of division given in Note C may be applied to the work of finding the G.C.M. The first two of the four given examples would then appear thus:—

3696	10080	2576	8848
1008	2688	336	1120
336	672		112

Note I. Page 75.

Sometimes it is required to multiply or divide, retaining only a certain number of decimal places in the product or quotient. The following are examples of this:—Multiply 827·6395 by ·17842, and divide 639·7218 by 329, and 171·963 by 2·835, in each case retaining three decimal places in the result.

827·6395	329)639·7218(1·944	2·835)171·963(60·6571
24871	3107	18630
827639	146	1620
579348	14	202
66211	1	4
3310		1
166		
147·6674		

The results are, therefore, 147·667; 1·94; 60·657.

In either case the decimals should be found to one more than the required number of places, and in multiplication the unit figure of the multiplier, or its place, if there be none, should be put under the decimal place of the multiplicand (in this case the fourth) to which the product is required to be accurate. The figures of the multiplier are to be put down in reverse order, and in the multiplication by each of them a beginning is to be made with the figure immediately above. In division the work proceeds in the ordinary way until we arrive at a point such that we are as many quotient figures distant from the last decimal place required as there are figures in the divisor. We then, instead of bringing down any more figures from the dividend, imagine one taken off from the right of the divisor every time. In both multiplication and division allowance must be made for the numbers which would have been carried if the multiplication had been entirely worked out.

These methods of contracted multiplication and division are open to one very grave objection, which almost entirely destroys their practical value. If they are used, the proof by casting out the nines becomes inapplicable, and it is in general better to use the longer form than to give up the power of employing this useful test.

NOTE K. PAGE 130.

The following method of calculating Compound Interest is based upon the expansion of the equation $\left(1 + \frac{R}{100}\right)^T$ into a series by the aid of the binomial theorem. It is useful, as it enables us to approximate rapidly to a correct result when a table of logarithms is not at hand. Let it be required to find the compound interest on £732 19*s.* 6*d.* for 23 years at 4 per cent. Here $\frac{R}{100}$ is $\frac{4}{100}$ or $\frac{1}{25}$. Let the principal be multiplied by $23 \times \frac{1}{25}$. Let this first product be multiplied by $\frac{22}{2} \times \frac{1}{25}$, forming a second product, which must be multiplied by $\frac{23}{3} \times \frac{1}{25}$ for a third product, this again by $\frac{24}{4} \times \frac{1}{25}$, and so on. The sum of these products will be the compound interest, which we can thus determine to any required degree of accuracy. Now £732 19*s.* 6*d.* = £732·975.

$$
\begin{array}{rl}
732{\cdot}975 \times \frac{23}{1} \times \frac{1}{25} = & 674{\cdot}337 \\
674{\cdot}337 \times \frac{22}{2} \times \frac{1}{25} = & 296{\cdot}70828 \\
296{\cdot}70828 \times \frac{21}{3} \times \frac{1}{25} = & 83{\cdot}07831 \\
83{\cdot}07831 \times \frac{20}{4} \times \frac{1}{25} = & 16{\cdot}61566 \\
16{\cdot}61566 \times \frac{19}{5} \times \frac{1}{25} = & 2{\cdot}52558 \\
2{\cdot}52558 \times \frac{18}{6} \times \frac{1}{25} = & {\cdot}30306 \\
{\cdot}30306 \times \frac{17}{7} \times \frac{1}{25} = & {\cdot}02944 \\
{\cdot}02944 \times \frac{16}{8} \times \frac{1}{25} = & {\cdot}00235 \\
{\cdot}00235 \times \frac{15}{9} \times \frac{1}{25} = & {\cdot}00016 \\
\hline
 & 1073{\cdot}59984
\end{array}
$$

The compound interest is therefore £1073·6, or £1073 12*s.* nearly.

The applicability of this method to mental calculation is fully explained in a pamphlet published by the Institution of Civil Engineers, containing a report of two lectures on Mental Calculation, delivered at meetings of that society by G. P. Bidder, Esq., Vice-President. The subject of these lectures is one which scarcely comes within the scope of the present work, and the reader is therefore referred to them as affording the soundest and most valuable information on this branch of arithmetic. At the same time they touch upon points which, though bearing upon Mental Calculation, have reference also to arithmetic generally. The subject of the present note is an example of this, and the next note

is merely a short statement of what will be found more fully explained in the lectures referred to.

Note L. Page 158.

A perfect square number must end in 1, 4, 5, 6, or 9, and the last digit but one may be any even digit before the figures 1, 4, 5, and 9, and any odd digit before 6. A perfect cube number may end in any combination of two digits whatever. In general there are four endings of two figures which will give the same last two figures in the square. Thus, numbers ending in 07, 43, 57, and 93, all have squares ending in 49; and numbers ending in 26, 24, 76, and 74, all have squares ending in 76. In the particular case where the number ends in 5 the square must end in 25, but 05, 45, 55, and 95 give 025 in the square; 15, 35, 65, and 85, give 225; and 25 and 75 give 625.

In general there are two endings of two figures which will give the same last two figures in an even cube. Thus numbers ending in 34 and 84 have cubes ending in 04, and numbers ending in 42 and 92 have cubes ending in 88. For an odd cube there is only one such ending. Thus numbers must end in 39 if the cube ends in 19, and in 37 if the cube ends in 53. In the particular case where the number ends in 5 the cube must end in 5; but 05, 25, 45, 65, and 85, give cubes ending in 25, while 15, 35, 55, 75, and 95, give cubes ending in 75.

Note M. Page 163.

When the common ratio is less than 1 the terms of a geometrical series continually diminish, and may, by taking a sufficient number of terms, be made as small as we please, and consequently 0 is the limiting value of the last term of such a series, when the number of terms is indefinitely increased. We may therefore apply the general rule to the case where a diminishing series is continued for ever. As the last term is nothing, the sum of the series must be the first term divided by the difference between the common ratio and unity. Thus the sum of the infinite series 14, 12, $10\frac{2}{7}$, &c., is $14 \div (1-\frac{6}{7})$ or 98. The sum of $1+\frac{1}{3}+\frac{1}{9}+$ &c., to infinity is $1 \div (1-\frac{1}{3}) = 1\frac{1}{2}$. A good idea may be obtained of the way in which the sums of such series approach a limit, by taking a strip of paper, and cutting off it lengths in G. P., as for example, 3 inches, 2 in., $1\frac{1}{3}$ in., and so on, and measuring to find how much has been cut off altogether.

Note N. Page 169.

It may, at first sight, seem strange that *any number whatever* when raised to the power of 0 should be equal to 1. The difficulty is however removed by considering this as an instance of a limiting value.

Take any number, say 40000, and, by successive applications of the rule for square root, find its second, fourth, eighth, sixteenth, &c., root. In this way we should find $(40000)^{\frac{1}{64}} = 1 \cdot 1801$; $(40000)^{\frac{1}{512}} = 1 \cdot 0209$; $(40000)^{\frac{1}{4096}} = 1 \cdot 0026$; $(40000)^{\frac{1}{32768}} = 1 \cdot 0003$. Here the root continually approaches unity, as the fractional index becomes smaller, and may, by continuing the process far enough, be made as near to unity as we please. Hence in the limit $(40000)^{0} = 1$.

Note O. Page 172.

The answers to this question and to the similar one, No. 10 of Ex. 44, are not strictly correct, as the equation $A = P\left(1 + \frac{R}{100}\right)^{T}$ does not give the amount at compound interest perfectly correctly when T is some number of intervals, and a fraction of an interval besides. This inaccuracy is owing to there being a Simple Interest calculation required to determine what is due for the fraction of an interval, and this would follow a different law from that of the equation. The effect of using this slightly inaccurate method is to make the amount at compound interest a little too small, and similarly in the case of present value to make the result obtained a little too large. To give an idea of the extent of this error we will take the case of £1000 at 5 per cent. compound interest for $2\frac{1}{2}$ years. Here, according to the correct plan, the amount for 2 years being 1102·5, the simple interest on this for $\frac{1}{2}$ year would be added, making 1130·0625, or £1130 1*s.* 3*d.*, while the amount calculated from the expression $1000 \times (1 \cdot 05)^{\frac{5}{2}}$ would be £1129 14*s.* 6*d.* The error therefore arising from this mode of calculation is very trifling, even where, as in this instance, a case has been selected, giving a comparatively large amount of inaccuracy. Applying the above explanation to the example worked out on page 172, we find from dividing log 5 by log (1·04) that the time is between 41 and 42 years. In 41 years a principal of £1 would increase to £4·99304, thus leaving £·00696 to be made up by simple interest for a fraction of a year. The time requisite for this would be $\frac{\cdot 0096}{4 \cdot 99304} \times \frac{100}{4}$ or ·0348 years. Hence, the correct answer would be 41·0348 years, or somewhat less than that previously obtained. In Ex. 44, No. 10, the time is first found to be between 36 and 37 years. A time of 36 years gives £5·7918 for the amount of £1, leaving ·2082 to be accounted for, and this is due to simple interest for ·7189 of a year, making the true answer 36·7189, instead of 36·7227.

ANSWERS TO EXAMPLES.

Ex. 1.

(1) 29; 86; 57.
(2) 354; 519; 608.
(3) 2511; 8796; 13009.
(4) 24902; 389746.
(5) 15,643528; 72,001004.
(6) 3002,000400,000017.

(7) Fifty-six; seventy-three; eighty-five. (8) Seven hundred and twenty-five; nine hundred and thirteen; Eight hundred and four.

(9) Four thousand, seven hundred and forty-two; Seven thousand and six; Twenty-three thousand, five hundred and eighty-nine.

(10) Fifty-one thousand, nine hundred and seventy-six. Two hundred thousand, three hundred and six.

(11) Forty-eight millions, two hundred and seventy-five thousand, three hundred and nineteen. Six hundred and seventy-five millions and thirty-two.

(12) Five billions, one hundred and seventy-nine thousand five hundred and eighty-four millions, three hundred and sixty-two thousand, eight hundred and fourteen. Nine hundred and forty billions, three hundred and twenty-thousand and five.

Ex. 2.

(1) 929.
(2) 242713.
(3) 332685.
(4) 596992.
(5) 93396.
(6) 319133.
(7) 417.
(8) 2678.
(9) 484192.
(10) 104763.
(11) 35108.
(12) 10889.
(13) 2.
(14) 29.
(15) 1.

Ex. 3.

(1) 48251; 39245; 603659.
(2) 295029; 630806; 1,180500.
(3) 27,776954; 33,738660.
(4) 27,313910; 484,935168.
(5) 51,981052; 437,326555.
(6) 5767,193817; 961,681644.

Ex. 4.

(1) 14387 rem. 1; 10309 rem. 7; 39640 rem. 2.
(2) 7011 rem. 3; 6471 rem. 1; 53262 rem. 7.
(3) 5287 rem. 1; 5029 rem. 3; 4806 rem. 11.
(4) 17820 rem. 22; 16977 rem. 7.
(5) 8148 rem. 75; 8267 rem. 90.
(6) 20150 rem. 221; 14665 rem. 26131.
(7) 6773 rem. 4696; 1432 rem. 1234.

Ex. 5.

(1) $17526\frac{2}{5}$; $5437\frac{1}{3}$; $7296\frac{3}{4}$.
(2) $874\frac{3}{11}$; $2470\frac{11}{14}$; $2995\frac{28}{31}$.
(3) $87\frac{10}{33}$; $738\frac{5}{14}$; $484\frac{1}{16}$.
(4) $\frac{10}{11}$; $7\frac{5}{16}$; $1\frac{7}{8}$; $2\frac{3}{7}$.
(5) $34\frac{1}{2}$; $94\frac{2}{3}$; 19; $413\frac{1}{3}$.
(6) 351; $104\frac{2}{5}$; $789\frac{1}{3}$; $1311\frac{3}{5}$.

(7) $\frac{4}{45}$; $\frac{9}{200}$; $\frac{5}{192}$; $\frac{1}{45}$.
(8) $\frac{19}{75}$; $\frac{29}{294}$; $\frac{181}{320}$; $\frac{177}{784}$.
(9) $1\frac{2}{3}$; $1\frac{3}{10}$; $\frac{11}{27}$; $\frac{16}{99}$.
(10) $\frac{5}{8}$; $\frac{3}{8}$; $\frac{13}{75}$; $\frac{117}{625}$.
(11) $7\frac{1}{2}$; 18; 80; $83\frac{1}{5}$.
(12) $20\frac{4}{7}$; 98; 210; $33\frac{1}{3}$.
(13) $53\frac{7}{25}$; $111\frac{1}{4}$; $110\frac{7}{10}$; $110\frac{10}{13}$.

Ex. 6.

(1) 658361; £1013 7*s.* $11\frac{1}{2}$*d.*
(2) 395684; 76958.
(3) £117106 15*s.*; 296381.
(4) 87625; 83847.
(5) 12016 h.g. 5*s.* 8*d.*; 796142.
(6) 4178 cr. 3*d.*; 96273.
(7) £340 16*s.* 3*d.*; £931 19*s.* $10\frac{3}{4}$*d.*
(8) 10919 g. 16*s.* 6*d.*; 10033 fl. 1*s.* $7\frac{1}{4}$*d.*
(9) 695327; 743681.
(10) 21996 qr. 7 lb. 1 oz.; 149 tons 9 cwt. 3 qr. 18 lb. 13 oz. 15 dr.
(11) 784259; 177 st. 3 lb. 10 oz. 1 dr.
(12) 785361; 168 lb. 1 oz. 11 dwt. 15 gr.
(13) 6 oz. 2 dr. 2 sc. 8 gr.; 34 lb. 6 dwt. 16 gr.
(14) 720608; 61 ld. 5 bus. 1 gal. 3 qt. 1 pt.
(15) 7654123; 2182 bus. 6 gal. 3 qt.
(16) 7582734; 15 miles 4 fur. 124 yd. 3 in.
(17) 63549; 538 po. 4 yd. 2 ft. 9 in.
(18) 673833; 55 lea. 7 fur. 168 yd. 1 ft.
(19) 693543; 588 sq. yd. 5 ft. 73 in.
(20) 624783; 310 roods 1194 yd.
(21) 25 perches 4 yd. 7 ft. 135 in. 1 sq. mile 393 acres 280 yd.
(22) 8647235; 199 cub. yd. 1 ft. 1359 in.
(23) 98543612; 14 yr. 95 days 17 hr. 31 min.
(24) 93276245; 62 days 1 hr. 46 min.

Ex. 7.

(1) £85 6*s.* 1*d.* (2) £107 2*s.* $4\frac{3}{4}$*d.*
(3) £185 6*s.* $5\frac{1}{2}$*d.*
(4) £32 7*s.* 1*d.*
(5) 35 tons 13 cwt. 1 qr. 25 lb.
(6) 2 cwt. 2 qr. 2 lb. 11 oz. 5 dr.
(7) 25 lb. 6 oz. 17 dwt. 7 gr.
(8) 48 bus. 1 qt.
(9) 25 ld. 3 qr. 3 bus.
(10) 27 lea. 2 m. 5 fur. 29 po.
(11) 133 mi. 7 fur. 68 yd. 1 ft.
(12) 23 po. 10 in.
(13) 52 po. 4 yd. 9 in.
(14) 16 ells 2 qr. 2 na.
(15) 87 sq. yd. 8 ft. 115 in.
(16) 126 acres 581 sq. yd.
(17) 22 acres 2 rd. $4\frac{1}{4}$ sq. yd.
(18) 174 cub. yd. 15 ft. 1386 in.
(19) 17 yr. 14 days 9 hr. 24 min.
(20) 22 wk. 1 d. 9 hr.
(21) £26 2*s.* $6\frac{3}{4}$*d.*
(22) £29 8*s.* $7\frac{1}{2}$*d.*
(23) 3 tons 16 cwt. 2 qr. 22 lb.
(24) 22 lb. 6 oz. 15 dr.
(25) 8 lb. 8 oz. 13 dwt. 12 gr.
(26) 1 oz. 5 dr. 2 sc. 15 gr.
(27) 1 bus. 3 pk. 1 gal. 1 qt.
(28) 1 mi. 6 fur. 131 yd. 2 ft.
(29) 2 po. 3 yd. 8 in.
(30) 6 sq. yd. 7 ft. 61 in.
(31) 13 acres 2 rd. 19 per.
(32) 3 per. 29 sq. yd. 6 ft. 88 in.
(33) 10 cub. yd. 22 ft. 1631 in.
(34) 6 yr. 177 d. 5 hr.
(35) £6 6*s.* $7\frac{1}{4}$*d.*

(36) £4 16*s*. $8\frac{3}{4}d$.
(37) £13 13*s*. $5\frac{3}{4}d$.
(38) £32 9*s*. $5\frac{3}{4}d$.
(39) 2 tons 12 cwt. 21 lb.
(40) 2 cwt. 2 qr. 23 lb. 7 oz.

Ex. 8.

(1) £23 9*s*. $7\frac{1}{2}d$.
(2) £53 0*s*. $1\frac{1}{2}d$.
(3) £81 16*s*. $1\frac{1}{4}d$.
(4) £166 3*s*. $4\frac{1}{2}d$.
(5) £280 11*s*. 3*d*.
(6) 3 tons 6 cwt. 3 qr. 1 lb. 13 oz.
(7) 25 lb. 6 oz. 7 dwt. 4 gr.
(8) 135 mi. 5 fur. 113 yd. 1 ft.
(9) 13 po. 3 yd. 2 ft. 6 in.
(10) £441 14*s*. $4\frac{1}{2}d$.
(11) £1096 3*s*. 5*d*.
(12) 209 ld. 2 qr. 3 bus.
(13) 371 yr. 99 d. 2 hr. 36 m.
(14) £369 11*s*. $6\frac{3}{4}d$.
(15) £360 13*s*. $6\frac{1}{2}d$.
(16) 2 tons 19 cwt. 11 lb. 11 oz.
(17) 319 sq. yd. 4 ft. 4 in.
(18) 6115 acres 2 rd.
(19) £250 8*s*. 10*d*.
(20) 8 tons 19 cwt. 2 qr. 6 lb. 11 oz.

Ex. 9.

(1) £5 4*s*. $3\frac{1}{4}d$.
(2) £6 12*s*. $3\frac{1}{4}d$.
(3) £8 2*s*. $6\frac{1}{4}d$.
(4) £7 19*s*. $3\frac{3}{4}d$.
(5) £9 7*s*. $6\frac{7}{10}d$.
(6) £2 5*s*. $7\frac{3}{14}d$.
(7) £2 18*s*. $2\frac{5}{8}d$.
(8) £1 7*s*. $9\frac{47}{48}d$.
(9) 1 qr. 24 lb. 5 oz.
(10) 3 lb. 3 oz. 18 dwt. 17 gr.
(11) 16 lb. $4\frac{7}{9}$ oz.
(12) 3 mi. 217 yd. 1 ft. 7 in.
(13) £3 1*s*. $6\frac{3}{8}d$.
(14) £9 16*s*. $6\frac{5}{96}d$.
(15) £4 1*s*. $4\frac{83}{140}d$.
(16) 2 qr. 24 lb. 2 oz. 12 dr.
(17) $7\frac{26}{77}$ in.
(18) 1 oz. 12 dwt. $10\frac{1}{2}$ gr.
(19) £4 8*s*. 11*d*.
(20) £13 2*s*. $3\frac{23}{38}d$.
(21) £14 9*s*. $3\frac{1}{2}d$.
(22) 4 gal. 1 qt.
(23) 56 days 16 hr. 2 m. $12\frac{1}{11}$ sec.
(24) 11 sq. yd. 3 ft. $114\frac{34}{59}$ in.
(25) £3 13*s*. $10\frac{1}{4}d$.
(26) 1 cub. yd. 19 ft. 531 in.
(27) £17 4*s*. $9\frac{183}{400}d$.
(28) 13*s*. $7\frac{11}{16}d$.
(29) 12*s*. $6\frac{1483}{2000}$d.
(30) £1 11*s*. $2\frac{127}{625}d$.

Ex. 10.

(1) 5. (2) 13. (3) 43.
(4) 81. (5) 113. (6) 149.
(7) 17. (8) 4 cwt. 7 lb. 10 oz.
(9) 3 sq. yd. 3 ft. 23 in.
(10) 18 sq. yd. 3 ft. 114 in.
(11) 16 sq. yd. 1 ft. 134 in.
(12) 49 sq. yd. 45 in.
(13) 39 sq. yd. 7 ft. 119 in.
(14) 54 sq. yd. 3 ft. 57 in.
(15) 38 yd. 1 ft. 1 in. (16) 2 yd.
(17) 11 yd. 2 ft. 9 in.
(18) 12 yd. 8 in.
(19) 3 cub. yd. 1 ft. 204 in.
(20) 46 cub. yd. 17 ft. 941 in.
(21) 27 sq. yd. 6 ft. 108 in.; 37 yd.
(22) £5 3*s*. $1\frac{1}{2}d$.
(23) 54 cub. ft. 1188 in.
(24) 1 ft. 6 in.

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(1) 77716263; 1231.
(2) £40 2*s.* 3*d.*; 1 ton 3 cwt. 2 qr. 6 lb. 7 oz.
(3) £77 3*s.* $5\frac{1}{2}$*d.*
(4) 3 roods 220 yd. (5) 103047.
(6) 287303161; $3359\frac{226}{229}$.
(7) 752; 120556 and 2*s.*
(8) 3 gallons 1 qt.; $3\frac{5}{7}$ pt.; 3 weeks.
(9) £2 10*s.* $3\frac{1}{4}$*d.* (10) 989411.
(11) 61. (12) 2*s.* $7\frac{1}{2}$*d.*
(13) 18 sq. yd. 7 ft. 18 in.; 25 yd. 2 in.
(14) 120240; 12 wk. 6 d. 18 hr. 41 m. 31 sec.
(15) £481 13*s.* $0\frac{1}{4}$*d.*; £13 16*s.* $10\frac{1}{2}$*d.*
(16) 7. (17) 40. (18) 10 days.
(19) 405. (20) £3 6*s.* 8*d.*
(21) 1 lb. 6 oz. 13 dwt. 22 gr.; 26 ld. 2 qr. 4 bush. 6 gal. 2 qt. 1 pt.
(22) £1 17*s.* $7\frac{1}{2}$*d.*
(23) 1 ton 6 cwt. 2 qr. 13 lb. 7 oz. 6 dr.; 6 yr. 284 d. 2 hr. 36 min. 15 sec.
(24) £903 15*s.*; £12 1*s.*
(25) 14 yd. and 7 yd.
(26) 11 acres 1204 sq. yd. 4 ft.
(27) £1. (28) £16 6*s.* $11\frac{1}{2}$*d.*
(29) 23. (30) 432000.
(31) £12 18*s.* $9\frac{1}{2}$*d.*, £10 13*s.* $6\frac{1}{2}$*d.*, £7 12*s.* $2\frac{1}{2}$*d.*, £4 19*s.* $8\frac{1}{2}$*d.*
(32) $14\frac{1}{2}$ days.
(33) £242 13*s.* $10\frac{1}{2}$*d.*; £8 7*s.* $4\frac{1}{2}$*d.*
(34) £1 1*s.* (35) $102665\frac{1}{4}$.
(36) £1 13*s.* $3\frac{1}{2}$*d.* to each man, and 16*s.* $7\frac{3}{4}$*d.* to each woman.
(37) 2838; 48. (38) 10*s.* 8*d.*
(39) $11\frac{11}{19}$ seconds.
(40) 24 tons 6 cwt. 17 lb. 2 oz. 12 dr.
(41) 420. (42) £547 17*s.* 6*d.*
(43) 42 gallons.
(44) £600, £400, £300.
(45) 2*s.* $7\frac{1}{4}$*d.*
(46) 84 dozen at 3*d.* and 42 dozen at 4*d.*
(47) £2075 18*s.* 9*d.* (48) 1800.
(49) £16 0*s.* 10*d.*
(50) 367 sq. yd. 4 ft. 16 in.

Ex. 11.

(1) 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167.
(2) 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096; 3, 9, 27, 81, 243, 729, 2187, 6561; 5, 25, 125, 625, 3125, 15625; 7, 49, 343, 2401; 11, 121, 1331, 14641.
(3) 169, 289, 361, 529, 841, 961.
(4) $2^2 \times 3 \times 7$. (5) $2^5 \times 3$.
(6) $2^3 \times 3^2 \times 5$. (7) $2^4 \times 3 \times 7$.
(8) $2^3 \times 3^3 \times 5 \times 11$.
(9) $2^4 \times 59$. (10) $2^2 \times 3^4 \times 7$.
(11) $2^3 \times 3 \times 5 \times 47$.
(12) $3 \times 5^2 \times 11 \times 19$.
(13) $11 \times 13 \times 37$.
(14) $7^2 \times 11 \times 23$.
(15) $2 \times 3 \times 5 \times 7 \times 11 \times 13$.
(16) $7^2 \times 11^2$. (17) $3^3 \times 5 \times 83$.
(18) $2^3 \times 3^2 \times 5 \times 11 \times 13$.
(19) $2^{12} \times 17$. (20) prime, 17.
(21) prime, 19. (22) 19×23.
(23) prime, 23. (24) prime, 23.
(25) 23×29. (26) $3 \times 17 \times 43$.
(27) $2 \times 17 \times 139$.
(28) $7 \times 23 \times 31$.

Ex. 12.

(1) 18. (2) 75. (3) 13. (10) 1760. (11) 1728.
(4) 8. (5) 125. (6) 14. (12) 4096. (17) 28. (18) 45.
(7) 137. (8) 259. (9) 37. (19) 25. (20) 2.

Ex. 13.

(1) 72. (2) 60. (3) 2520. (12) 5460. (13) 4862.
(4) 3465. (5) 360. (6) 1080. (14) 2520. (15) 232792560.
(7) 3960. (8) 17820. (9) 504. (16) 14549535. (17) 5040.
(10) 12600. (11) 1296. (20) 52 minutes.

Ex. 14.

(1) $\frac{7}{10}$. (2) $\frac{7}{13}$. (3) $\frac{16}{45}$. (13) $\frac{113}{119}$. (14) $\frac{395}{536}$. (15) $\frac{17}{29}$.
(4) $\frac{4}{5}$. (5) $\frac{7}{9}$. (6) $\frac{25}{33}$. (16) $\frac{11}{14}$. (17) $\frac{23}{24}$. (18) $\frac{42}{67}$.
(7) $\frac{24}{25}$. (8) $\frac{6}{11}$. (9) $\frac{4}{5}$. (19) $\frac{19}{27}$. (20) $\frac{601}{1394}$.
(10) $\frac{44}{117}$. (11) $\frac{21}{32}$. (12) $\frac{2}{3}$.

Ex. 15.

(1) $\frac{39}{320}$. (2) $\frac{11}{32}$. (3) $\frac{1}{7}$. (13) $\frac{1}{1125}$. (14) $1\frac{9}{11}$. (15) $\frac{6}{37}$.
(4) $\frac{4}{15}$. (5) $\frac{3}{16}$. (6) 2. (16) $\frac{37}{54}$. (17) $14\frac{2}{3}$. (18) $3\frac{3}{5}$.
(7) $\frac{216}{385}$. (8) 54. (9) $9\frac{3}{7}$. (19) $1\frac{1}{5}$. (20) $4\frac{4}{11}$.
(10) 1. (11) $2\frac{2}{11}$. (12) $\frac{49}{198}$.

Ex. 16.

(1) $3\frac{53}{72}$. (2) $2\frac{91}{252}$. (3) $2\frac{143}{200}$. (19) $7\frac{1}{2}$. (20) $2\frac{5}{12}$. (21) $\frac{31}{36}$.
(4) $21\frac{29}{220}$. (5) $20\frac{83}{315}$. (6) $21\frac{19}{99}$. (22) $6\frac{5}{7}$. (23) $6\frac{5}{9}$. (24) $1\frac{3}{5}$.
(7) 1. (8) $27\frac{11}{120}$. (9) $35\frac{19}{30}$. (25) $13\frac{3}{7}$. (26) 0.
(10) $\frac{3}{40}$. (11) $\frac{9}{20}$. (12) $\frac{10}{143}$. (27) $\frac{17}{25}$ is the greatest, and $\frac{71}{105}$ the least.
(13) $\frac{41}{300}$. (14) $4\frac{7}{20}$. (15) $2\frac{1}{14}$.
(16) $1\frac{31}{40}$. (17) $3\frac{1}{2}$. (18) $1\frac{5}{8}$.

Ex. 17.

(1) 77·599. (2) 80·901. (7) 175·006. (8) 516·9924.
(3) 114·29. (4) 2096·897. (9) 172·463. (10) 14·427.
(5) 4136·6314. (6) 3·877.

Ex. 18.

(1) 38·08. (2) 157·5. (15) ·0016. (16) 6250.
(3) ·08843. (4) 120·4645. (17) 1080. (18) ·0625.
(5) 32. (6) ·003821967. (19) 84·15. (20) 404·5625.
(7) 6·72888. (8) 145287. (21) ·005976. (22) 235125.
(9) ·761219. (10) 800. (23) ·0037432. (24) 10278·125.
(11) ·0084. (12) 350·064. (25) 4937000. (26) 88647.
(13) 14·58. (14) ·00036288.

Ex. 19.

(1) $\cdot 40625$. (2) $\cdot 00975$. (3) $\cdot 033125$. (4) $\cdot 8112$. (5) $\cdot 115$. (6) $\cdot 018359375$. (7) $16\cdot 225$. (8) $11\cdot 8375$. (9) $14\cdot 01975$. (10) $\cdot 6\dot{1}$. (11) $\cdot\dot{5}1\dot{8}$. (12) $\cdot 361\dot{4}8\dot{6}$.

(13) $\cdot 059\dot{8}5714\dot{2}$. (14) $\cdot 948\dot{1}707\dot{3}$. (15) $25\cdot\dot{1}88\dot{1}$. (16) $37\cdot 5208\dot{3}$. (17) $4\cdot 065\dot{9}\dot{0}$. (18) $\cdot\dot{3}6609\dot{3}$. (19) $41\cdot 66\dot{7}808219\dot{1}$. (20) $\cdot 01\dot{1}5384\dot{6}$.

Ex. 20.

(1) $\frac{7}{32}$. (2) $\frac{11}{6250}$. (3) $5\frac{9}{125}$. (4) $4\frac{927}{390625}$. (5) $\frac{55}{128}$. (6) $3\frac{15}{16}$. (7) $\frac{1679}{20000}$. (8) $\frac{509}{640}$. (12) $\frac{13}{22}$. (13) $12\frac{11}{24}$. (14) $8\frac{4}{15}$.

(15) $7\frac{5}{37}$. (16) $\frac{32}{75}$. (17) $13\frac{7}{88}$. (18) $\frac{79}{505}$. (19) $10\frac{137}{275}$. (20) $\frac{17}{540}$. (21) $\frac{139}{271}$. (22) $13\frac{677}{11950}$. (23) $21\frac{127}{208}$. (24) $\frac{379}{1025}$.

Ex. 21.

(1) $\cdot\dot{6}$. (2) $\cdot 88\dot{7}26\dot{3}$. (3) $2\cdot 871690\dot{0}4\dot{5}$. (4) $10\cdot 665\dot{1}29\dot{0}$. (5) $\cdot 25\dot{0}5\dot{0}$. (6) $3\cdot 6443\dot{9}81\dot{4}$. (7) $\cdot\dot{2}47\dot{5}$. (8) $\cdot 09375$. (9) $\cdot 2\dot{7}$. (10) $\cdot 84$. (11) $\cdot\dot{1}140\dot{3}$. (12) 1. (13) 22. (14) $\cdot 001\dot{0}\dot{9}$. (15) $\cdot\dot{6}$.

(16) $\cdot 1\dot{6}\dot{9}$. (17) 1 and 3; 3 and 5. (18) 3 and 8; 4 and 5. (19) 1 and 10; 1 and 20. (20) 3 and 18; 0 and 120. (21) $10\cdot 200047$. (22) $16\cdot 919193$. (23) $2\cdot 374948$.

Ex. 22.

(1) 1*s.* $10\frac{1}{2}$*d.*; 11*s.* 8*d.*; 10*s.* $7\frac{1}{2}$*d.*

(2) £1 2*s.* $2\frac{2}{3}$*d.*; £2 11*s.* $2\frac{41}{44}$*d.*; £3 17*s.*

(3) 6 cwt. 3 qr. 24 lb.; 1 ton 9 cwt. 1 qr. 9 lb. 10 oz.

(4) 6 lb. 1 oz. 6 dwt. 16 gr.; 4 lb. 3 oz. 11 dwt. $6\frac{1}{9}$ gr.

(5) £11 13*s.* $11\frac{1}{4}$*d.*; 1 rood $775\frac{1}{2}$ yd.

(6) 3 mi. 3 fur. $41\frac{1}{4}$ yd.; 1 qr. 5 bus. 2 qt.

(7) 6 wk. 4 d. 15 hr. 34 m. 24 s.; 6 cub. ft. 324 in.

(8) £1 4*s.* $4\frac{1}{2}$*d.*; 12 hr. 26 m. 40 s.

(9) £13 15*s.*; 1 cwt. 2 qr. 8 lb.

(10) 5*s.* 7*d.*; £1 8*s.* $7\frac{3}{4}$*d.*

Ex. 23.

(1) $\frac{65}{192}$; $\frac{1}{28}$. (2) $2\frac{9}{25}$; $\frac{23}{90}$. (3) $\frac{9}{112}$; $\frac{28}{45}$. (4) $2\frac{19}{40}$; $\frac{25}{32}$. (5) $7\frac{1}{9}$; $\frac{81}{1600}$. (6) $1\frac{1}{5}$; 1.

(7) $\frac{8}{11}$; $\frac{1}{27}$. (8) $\frac{16}{99}$; $\frac{11}{42}$. (9) $10\frac{43}{79}$; $\frac{81}{484}$. (10) $\frac{1}{21}$; $2\frac{1}{2}$.

Ex. 24.

(1) £1 7*s.* 6*d.*; 13*s.* 4*d.*

(2) 6*s.* 3*d.*; £1 19*s.* $4\frac{1}{2}$*d.*

(3) $5\cdot 625$*d.*; £1 12*s.* 7*d.*

(4) 25 lb. 2 oz.; 4 gal. 3 qt.

(5) 2 lb. 8 oz. 6 dwt. $8\cdot 256$ gr.; $4\cdot 224$*d.*

(6) 11 wk. 2 d. 16 hr. 19 m. 12 s.; 1 ton 17 cwt. 1 qr. 14 lb. 14 oz.

(7) 6 mi. 1 fur. 197·12 yd.; 1 rood 27 per. $22\frac{1}{4}$ yd.

(8) 19 cub. yd. 10 ft. 752·544 in.; 7*s*.

Ex. 25.

(1) ·7875; 1·5125.

(2) ·3125; ·0625.

(3) ·6810546875; 109·375.

(4) ·575; 1·625.

(5) ·0004125; ·4625.

(6) 145·525; ·8078125.

(7) 7·53125; ·655859375.

(8) 8·75; ·625.

Ex. 26.

(1) 9*s*. 10*d*.; £3 6*s*. $1\frac{3}{4}$*d*.

(2) 19*s*. $1\frac{1}{2}$*d*.; 2*s*. $10\cdot1\dot{6}$*d*.

(3) 3 fur. 53 yd. 1 ft.; 5 d. 16 hr. 53 m. 20 s.

(4) 4 sq. ft. 82 in.; 1 day.

(5) $2\cdot\dot{3}$*d*.; £2 4*s*. $10\cdot6\dot{3}$*d*.

(6) $\cdot42708\dot{3}$; $\cdot4\dot{0}1234567\dot{9}$.

(7) $\cdot533482\dot{1}4285\dot{7}$; $2\cdot3\dot{7}804\dot{8}$.

(8) $\cdot\dot{4}3\dot{2}$; $\cdot\dot{4}087591\dot{2}$.

(9) $\cdot\dot{9}7530864\dot{1}$; $\cdot\dot{4}7186\dot{1}$.

(10) $\cdot37\dot{4}2857\dot{1}$; $\cdot000\dot{0}3\dot{7}$.

(11) 6 and 6.

(12) 0 and 24.

(13) 3 and 1.

(14) 1 and 2.

(15) 2 and 198.

(16) 0 and 30.

Ex. 27.

(1) £1127 8*s*. $1\frac{1}{2}$*d*.; £5015 5*s*. 4*d*.

(2) £2780 6*s*. $6\frac{1}{2}$*d*.; £2173 18*s*. $7\frac{3}{4}$*d*.

(3) £89 17*s*. 3*d*.; £2600 12*s*. 11*d*.

(4) £13893 15*s*. $4\frac{1}{4}$*d*.; £2699 11*s*. 5*d*.

(5) £4586 17*s*. $8\frac{1}{4}$*d*.; £25120 9*s*. 3*d*.

(6) £416 14*s*. $1\frac{1}{2}$*d*.; £1919 2*s*. $10\frac{3}{4}$*d*.

(7) £69169 18*s*. $4\frac{1}{2}$*d*.; £55 10*s*. $1\frac{1}{2}$*d*.

(8) 181 tons 5 cwt. 16 lb.; 2540 lb. 11 oz. 5 dwt.

(9) 7 ld. 3 qr. 6 bus. 5 gal. 3 qt. 1 pt.; 10 mi. 3 fur. 101 yd. 8 in.

(10) 33 yr. 2 d. 23 hr. 38 m. 21 sec.; 3799 mi. 3 fur. 29 po.

(11) 12 ton 14 cwt. 10 lb. 6 oz. 6 dr.; 3978 sq. yd. 8 ft. 40 in.

Ex. 28.

(1) £124 1*s*. $1\frac{1}{2}$*d*.

(2) £419 5*s*. $7\frac{1}{8}$*d*.

(3) £14378 6*s*. 4*d*.

(4) £218 8*s*. $6\frac{7}{12}$*d*.

(5) £393 5*s*. 10*d*.

(6) £345 0*s*. $8\frac{7}{16}$*d*.

(7) £3 0*s*. $5\frac{21}{32}$*d*.

(8) £19 2*s*. $1\frac{5}{8}$*d*.

(9) £27 6*s*. $4\frac{1}{4}$*d*.

(10) £161 9*s*. $8\frac{1}{2}$*d*.

(11) £460 3*s*. $7\frac{1}{8}$*d*.

(12) £522 4*s*. 6*d*.

(13) £619 3*s*. $0\frac{1}{2}$*d*.

(14) £127 16*s*. $1\frac{3}{4}$*d*.

Ex. 29.

(1) £79 1*s*. 9*d*.

(2) £20 5*s*. 2*d*.

(3) 594.

(4) $58\frac{2}{3}$.

(5) £31 10*s*.

(6) 64 ft.

(7) 7 hr. 4 m. 20 s. A. M.

(8) 31 minutes past 1.

(9) £5 8*s*. $5\frac{67}{73}$*d*.

(10) £16 6*s*. 8*d*.

(11) 3*s*. $7\frac{1}{2}$*d*.

Ex. 30.

(1) 36.
(2) 1000.
(3) 288.
(4) 15.
(5) $\frac{1}{275625}$.
(6) $15\frac{5}{12}d.$
(7) $10\frac{1}{2}$ hr.
(8) 2 wk. $2\frac{1}{4}$ d.

Ex. 31.

(1) £60 14*s.* 6*d.*
(2) £10 12*s.* $2\frac{6}{17}d.$
(3) 11 seconds.
(4) £3 12*s.*
(5) 33·95 feet.
(6) 87.
(7) £27 15*s.*
(8) £581 10*s.* 7*d.*
(9) $20\frac{2}{3}$ hours.
(10) 1*s.*
(11) £3 15*s.*
(12) 1609·306.
(13) 2250.
(14) £1 1*s.* $10\frac{1}{2}d.$
(15) £200.
(16) 312 days.
(17) $157\frac{1}{2}$.
(18) £6 12*s.*
(19) £1 4*s.*
(20) $6\frac{1}{2}d.$

Ex. 32.

(1) 8.
(2) 5.
(3) £26 13*s.* 4*d.*
(4) £427 10*s.*
(5) £16 18*s.* 9*d.*
(6) 30.
(7) 30 days.
(8) £7 17*s.* 6*d.*
(9) 1 lb.
(10) £1375.
(11) 96.
(12) 21.

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(1) 3960; 499.
(2) $6\frac{965}{1092}$; $7\frac{23}{60}$; $\frac{2}{21}$; 8*s.*
(3) $\frac{43}{50}$; $\frac{9}{11}$.
(4) $\frac{3}{80}$; $2\frac{4}{125}$; $\frac{19}{37}$.
(5) ·8125; $\cdot 0287\dot{1}6\dot{2}$;
20, 2000, ·0002, 2.
(6) $\frac{145}{192}$; 8 cwt. 1 qr. 3 lb. 1 oz.
(7) 33.
(8) £9 17*s.* 3*d.*
(9) 495; 2880.
(10) $14\frac{103}{252}$; $3\frac{433}{1008}$.
(11) 3·44; $11\cdot\dot{0}0\dot{3}$; $1\frac{3}{16}$; $\frac{233}{370}$.
(12) $\frac{11}{15}$.
(13) ·046152; ·0000641.
(14) $\frac{11}{960}$; 6 ton 18 cwt. 1 qr.
(15) ·00003125; 9*s.* $9\frac{3}{8}d.$
(16) 20 qr. 5 bus.
(17) £520.
(18) £15 1*s.* $5\frac{1}{4}d.$
(19) 15.
(20) 176; 360.
(21) $\frac{1}{4}$; $\frac{1024}{1125}$; $\frac{83}{113}$.
(22) £55 2*s.* $2\frac{2}{5}d.$
(23) $13\frac{261}{280}$; $4\frac{14}{15}$; $17\frac{1071}{10100}$.
(24) 26 years.
(25) 3 years; 25 days.
(26) £3600.
(27) 64 bus. 2 pk. 1 gal.; $\frac{17}{1440}$.
(28) 28 sq. yd. 5 ft. 117 in.; $41\frac{1}{4}$;
£5 10*s.*
(29) 2 qr. 9·66 lb.; ·007421875.
(30) ·125.
(31) 45738.
(32) 50 cubic feet.
(33) £6000.
(34) 7*s.* $3\frac{1}{9}d.$
(35) £3 10*s.* 4*d.*
(36) 13 ft. $5\frac{1}{4}$ in.
(37) £2 1*s.* $1\cdot\dot{9}8701\dot{2}$; ·5.
(38) £29 4*s.* 8·87*d.*; 1 mi. 6 fur.
120 yd.
(39) $\frac{9}{40}$.
(40) 3·1415927.
(41) £2 7*s.* $10\frac{3}{4}d.$
(42) 4 P.M. Oct. 1st.; 5 hr. 30 m.
A.M.
(43) £1 11*s.* 6*d.*
(44) $1\cdot1\dot{2}8048\dot{7}$.
(45) 15 mi. 5 fur.
(46) 2 and 6; 1 and 40; 7 and 4;
8 and 1560.
(47) 6 cwt.
(48) 61 miles.
(49) 2·7182818.
(50) 18 cwt. 2 qr. 7 lb.

Ex. 33.

(1) £72 10*s*. (2) £40 11*s*. $9\frac{3}{8}d$. (6) £1341. (7) £155 5*s*. $1\frac{7}{20}d$.
(3) £149 12*s*. $1\frac{3}{16}d$. (8) £316 9*s*. $11\frac{154}{365}d$.
(4) £129 2*s*. $8\frac{1}{2}d$. (9) £11 10*s*. $1\frac{47}{73}d$.
(5) £96 0*s*. $2\frac{1}{4}d$. (10) £81 11*s*. $11\frac{1}{4}d$.

Ex. 34.

(1) $4\frac{7}{12}$ years. (2) $3\frac{1}{2}$. (7) 6. (8) £1910 15*s*.
(3) $3\frac{1}{3}$ years. (4) £569 5*s*. 5*d*. (9) 4 years 40 days.
(5) $5\frac{1}{2}$. (6) 2 years. (10) $9\frac{1}{6}$.

Ex. 35.

(1) £13 2*s*. $3\frac{6}{7}d$. (2) £1 6*s*. 5*d*. (7) £17 8*s*. (8) 5.
(3) £402 10*s*. (4) $\frac{9}{406}$. (9) 15*s*. $1\frac{1}{2}d$.
(5) £5 6*s*. $11\frac{37}{73}d$. (6) 9*s*. 6*d*. (10) 2 years.

Ex. 36.

(1) £1228 5*s*. (2) £276 9*s*. 3*d*. (7) £883 1*s*. $0\frac{3}{40}d$.
(3) £714 12*s*. $2\frac{38}{125}d$. (8) £3 19*s*. $8\frac{4}{25}d$.
(4) £51 0*s*. 3·44*d*. (9) £683 18*s*. $2\frac{4}{5}d$.
(5) £468 15*s*. (6) £1367 3*s*. 9*d*. (10) £153 6*s*. 8*d*.

Ex. 37.

(1) £2629 13*s*. 9*d*. (8) £1350. (9) $5\frac{5}{11}$.
(2) £1333 6*s*. 8*d*. (10) £276. (11) $13\frac{4}{27}$ and $5\frac{5}{9}$.
(3) $93\frac{1}{4}$. (4) £158 8*s*. $1\frac{1}{2}d$. (12) 93.
(5) £$92\frac{1}{8}$. (6) £75. (13) An increase of £7 10*s*.
(7) The 3 per cents.; £61182 15*s*. (14) 204.

Ex. 38.

(1) £85 6*s*. 3*d*. (2) £5. (8) £58 2*s*. 6*d*.
(3) £1900. (9) 8*d*. (10) £1 13*s*. 4*d*.
(4) £4948 9*s*. $0\frac{84}{97}d$. (11) 85·12 lb., 16·8 lb., 8·96 lb., 1·12 lb.
(5) $4\frac{1}{2}d$. (6) 35.
(7) £24. (12) 26·09, 16·87, 29·07 and 8·3.

Ex. 39.

(1) 23 canne 2·9 palmi. (7) $\frac{1}{10}$ per cent. cheaper in Paris, $\frac{1}{2}$ per cent. dearer in Hamburg.
(2) 308·68. (3) 4·047.
(4) 4729 francs 20 centimes.
(5) 50*d*. (6) $52\frac{3}{4}d$. (8) £14 6*s*. $7\frac{1}{4}d$.

Ex. 40.

(1) 336, 252, 609, 994. (6) £570.
(2) £1538 10*s*., £1031 14*s*., £724. (7) £120 6*s*. 3*d*., £154 13*s*. 9*d*., £182 16*s*. 3*d*., £207 3*s*. 9*d*.
(3) £656 5*s*., £1200, £990. (8) 107 : 105.
(4) 13 dwt. (5) $\frac{91}{513}$, $\frac{13}{57}$, $\frac{143}{513}$.

Ex. 41.

(1) 5879. (2) 8765. (5) 10·9087. (6) 8·59953.
(3) 12·397. (4) 212·49. (7) 117 yd. 2 ft. 3 in. (8) 34.

Ex. 42.

(1) 354. (2) 563. (6) 4·41332.
(3) 486·4. (4) 18·08295. (7) 57·8987 inches.
(5) 2·22398.

Ex. 43.

(1) 47. (2) 339. (7) $222\frac{1}{2}$. (8) $2\frac{14}{25}$.
(3) 272. (4) 165. (9) 1600 feet.
(5) 1536; 24570. (6) $\frac{4}{81}$; $53\frac{7}{9}$. (10) The former; £213 15*s.* 10*d.*

Ex. 44.

(1) 46·8751. (2) 23·7918. (9) £739 12*s.* 5*d.*
(3) ·0649117. (4) 447718·7. (10) 36·227.
(5) 36·759792. (6) 3·27236. (11) £694 7*s.* 8*d.*
(7) £868 12*s.* 5*d.* (12) £2077 1*s.* 3*d.*
(8) £587 13*s.* $4\frac{1}{2}$*d.*

MISCELLANEOUS EXAMPLES. Page 173.

(1) £76 19*s.* $11\frac{1}{4}$*d.* (2) £1 10*s.*
(3) £141 13*s.* 4*d.* (4) lose 25.
(5) £57 12*s.*
(6) 1683; 6·0285.
(7) £712, £578 10*s.*, £511 15*s.*, £333 15*s.*
(8) 5.
(9) $17\frac{7}{33}$; 161; 432; 7 cwt. 3 lb. $6\frac{34}{37}$ dr.; 787·$\dot{0}2\dot{7}$.
(10) $21\frac{9}{11}$ and $54\frac{6}{11}$ minutes past 4.
(11) 11 ft. (12) 14*s.* $4\frac{1}{2}$*d.*
(13) 56 lb., 35 lb., 21 lb.
(14) ·02478; ·00000224; 15·631$\dot{0}\dot{6}$.
(15) $7\frac{11}{30}$ days. (16) £37 8*s.*
(17) 67329; 18·02.
(18) £1 1*s.* (19) 168; $368\frac{116}{125}$.
(20) 24. (21) 12*s.*
(22) 23000 and 52500 gallons; loss to revenue £1568 15*s.*
(23) £39 1*s.*; £40 0*s.* $6\frac{3}{10}$*d.*; £35 10*s.*
(24) 5 seconds.
(25) £50, £75, £100, £125.
(26) $14\frac{53}{720}$; ·1712; ·38$\dot{4}$390$\dot{2}$; ·53$\dot{6}$2$\dot{9}$; ·007392.
(27) $98\frac{3}{4}$. (28) 1·294$\dot{2}$.
(29) $2^3 \times 3^3 \times 13 \times 59$; $5^2 \times 7 \times 23 \times 41$; $2 \times 3 \times 17 \times 19 \times 37$.
(30) $7\frac{1}{2}$ days.
(31) 11·6919; ·9353.
(32) 6*s.* 4*d.* (33) 2208 feet.
(34) £112 10*s.*, £150, £191 5*s.*, £234.
(35) £44 15*s.* 6*d.*; £147 5*s.* $1\frac{7}{25}$*d.*; £13 19*s.* $0\frac{4}{27}$*d.*
(36) 288; 574560. (37) 150.

(38) 26 lb. 4 oz.; ·005; £15 15*s.* 3*d.*
(39) 139 thalers $8\frac{3}{4}$ s.g.
(40) 1*s.* 3*d.* (41) $4\frac{1}{2}$ miles.
(42) $\frac{3}{4}$ inch. (43) $58\frac{2}{3}$ days.
(44) £546 2*s.* 8*d.*; £1092 5*s.* $3\frac{3}{4}$*d.*
(45) 55; 66; 78.
(46) 27·86; 47·84; ·19; 1·57; 22·54.
(47) 1*s.* 3*d.* (48) £99 13*s.* $11\frac{19}{40}$*d.*
(49) 4. (50) 1*s.* 1*d.*
(51) $3\frac{1}{2}$. (52) 656.
(53) ·0000002667; 3 yd. 2 ft. 3 in.
(54) £41 3*s.* 4*d.*
(55) $10\frac{10}{11}$ and $43\frac{7}{11}$ minutes past 5.
(56) £1737 10*s.* (57) ·286.
(58) 19 min. 49 sec.
(59) £90, £120, £135, £150.
(60) $12\frac{1}{2}$ minutes.
(61) £1302 1*s.* 8*d.* (62) 6.
(63) Every $17\frac{1}{3}$ minutes, at points distant $\frac{1}{3}$ of a round from each other.
(64) 52 minutes.
(65) $8\frac{2}{5}$ inches.
(66) 92, 1944; 891, 12023.
(67) £3 8*s.* 3*d.* (68) 98.
(69) 22 oz. (70) $42\frac{6}{7}$ per cent.
(71) 1·6189. (72) £3289.
(73) $3\frac{3}{4}$ days. (74) 2*s.* 8*d.*
(75) £2673; £7315.
(76) $1\frac{1}{8}$ miles; 9 and $8\frac{8}{11}$.
(77) £4400; £3547 10*s.*; £3372; £3200.
(78) $19140\frac{5}{8}$.
(79) 253·69 feet per second.
(80) £113.
(81) $2^3 \times 5 \times 11 \times 13 \times 19$; $2^{10} \times 47$; $2 \times 3 \times 7^2 \times 17 \times 23$.
(82) £67 1*s.* 8*d.*
(83) $52\frac{1}{2}$ minutes past 1.
(84) £3 17*s.* $10\frac{1}{2}$*d.*
(85) £29 1*s.* $4\frac{7}{8}$*d.*
(86) $5\frac{1}{2}$, $8\frac{1}{2}$, $11\frac{1}{2}$, $14\frac{1}{2}$.
(87) £177 2*s.*; £343 15*s.*; £384.
(88) 40. (89) £584 6*s.* 7*d.*
(90) £3573 13*s.* $4\frac{1}{2}$*d.*
(91) 13; $41\frac{7}{13}$ minutes past 3; $9\frac{3}{13}$ minutes, and $36\frac{12}{13}$ minutes past 7.
(92) 10 and 0; 12 and 3; 30 and 3.
(93) $66\frac{2}{3}$ minutes; $9123\frac{21}{25}$.
(94) 26 ft. 1·15 in.
(95) $1\frac{1}{2}$ miles; 6 minutes.
(96) £83 6*s.* 8*d.*
(97) 9 florins 50·7 kreutzers.
(99) $88\frac{3}{4}$ sq. yd.
(100) 360.

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